

MATH425

Quantum Field Theory

Homework Sheet 0

<https://math425.yannickulrich.com>

Issued: 5 December

Due: never

This bonus sheet is a collection of problems you may find useful. Please consider these questions in addition to the exercises in the lecture notes.

Note: the content covered here is not necessarily the same as in the exam.

Homework 1: Quantum harmonic oscillator

The Hamiltonian for the quantum harmonic oscillator is given by $\hat{H} = \frac{\hat{p}^2}{2m} + \frac{m\omega^2}{2}\hat{q}^2$.

- (2 Pts.) Using the fact that $[\hat{q}, \hat{p}] = i$, rewrite the Hamiltonian with the raising and lowering operators \hat{a} and \hat{a}^\dagger that have the commutation relations $[\hat{a}, \hat{a}^\dagger] = 1$.
- (2 Pts.) What are the commutation relation of the operator and $N = \hat{a}^\dagger \hat{a}$ with the Hamiltonian operator?
- (2 Pts.) With the help of the \hat{a} and \hat{a}^\dagger operators, construct the spectrum of energy eigenstates of the Hamiltonian starting from the vacuum state $|0\rangle$.
- (1 Pts.) Find corresponding energy levels?
- (1 Pts.) What is the energy level of the vacuum state $|0\rangle$? How would you interpret this as in implication for observable phenomena?

SOLUTION:

- a) The Hamiltonian for a quantum harmonic oscillator is

$$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega^2\hat{q}^2.$$

We define the raising and lowering operators \hat{a} and \hat{a}^\dagger as

$$\hat{q} = \sqrt{\frac{1}{2m\omega}}(\hat{a} + \hat{a}^\dagger), \quad \hat{p} = i\sqrt{\frac{m\omega}{2}}(\hat{a}^\dagger - \hat{a}).$$

Substituting these into the Hamiltonian, we find:

$$\hat{H} = \omega\left(\hat{a}^\dagger\hat{a} + \frac{1}{2}\right).$$

- b) The number operator is $\hat{N} = \hat{a}^\dagger\hat{a}$. The commutation relations are

$$[\hat{H}, \hat{a}] = -\omega\hat{a}, \quad [\hat{H}, \hat{a}^\dagger] = \omega\hat{a}^\dagger.$$

and for the number operator

$$[\hat{N}, \hat{a}] = -\hat{a}, \quad [\hat{N}, \hat{a}^\dagger] = \hat{a}^\dagger.$$

- c) Define the vacuum state $|0\rangle$ such that $\hat{a}|0\rangle = 0$. The excited states are obtained by applying \hat{a}^\dagger

$$|n\rangle = \frac{(\hat{a}^\dagger)^n}{\sqrt{n!}}|0\rangle.$$

- d) The energy levels of the oscillator are

$$E_n = \omega \left(n + \frac{1}{2} \right), \quad n = 0, 1, 2, \dots$$

- e) The energy of the vacuum state $|0\rangle$ is

$$E_0 = \frac{1}{2}\omega.$$

This is the zero-point energy, an inherent energy due to quantum fluctuations. It implies particles cannot have zero energy, contributing to phenomena such as the Casimir effect.

Homework 2: Harmonic oscillator and Euler-Lagrange equations The potential function of a one dimensional harmonic oscillator is given by

$$V(x) = \frac{1}{2}kx^2.$$

- a) (1 Pts.) Write the Lagrangian $L = T - V$, where T is the kinetic energy and V is the potential energy.
- b) (1 Pts.) Use the Euler-Lagrange equation to derive the equation of motion.
- c) (2 Pts.) Using the initial conditions $x(0) = 0$ and $\dot{x}(0) = v_0$, find the specific solution for $x(t)$.
- d) (2 Pts.) Write expressions for the kinetic energy $T(t)$, potential energy $V(t)$, and the total energy E of the system. Prove that the total energy is conserved over time.
- e) (2 Pts.) Suppose a damping force proportional to velocity is added to the system, such that the equation of motion becomes $m\ddot{x} + b\dot{x} + kx = 0$ where b is the damping coefficient. Solve the new equation of motion for the underdamped case (where $b^2 < 4mk$) and describe how the motion differs from the undamped case.

SOLUTION:

- a) The potential is $V(x) = \frac{1}{2}kx^2$ and the kinetic energy $T = \frac{1}{2}m\dot{x}^2$. Therefore, the Lagrangian is

$$L = T - V = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}kx^2.$$

- b) Using the Euler-Lagrange equation

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{x}} \right) - \frac{\partial L}{\partial x} = 0.$$

we find

$$m\ddot{x} + kx = 0.$$

- c) With initial conditions $x(0) = 0$ and $\dot{x}(0) = v_0$, the solution to $m\ddot{x} + kx = 0$ is

$$x(t) = \frac{v_0}{\omega} \sin(\omega t), \quad \text{where } \omega = \sqrt{\frac{k}{m}}.$$

- d) The kinetic energy is $T(t) = \frac{1}{2}m\dot{x}^2$ and potential energy is $V(t) = \frac{1}{2}kx^2$. Using $x(t)$ from above, we find that the total energy $E = T + V$ remains constant over time, indicating conservation of energy.

- e) For the damped equation $m\ddot{x} + b\dot{x} + kx = 0$ with $b^2 < 4mk$, the solution is

$$x(t) = e^{-\gamma t} (C_1 \cos(\omega' t) + C_2 \sin(\omega' t)).$$

where $\gamma = \frac{b}{2m}$ and $\omega' = \sqrt{\omega^2 - \gamma^2}$. This represents oscillatory motion with a decreasing amplitude due to damping.

To verify, substitute $x(t) = e^{-\gamma t} (C_1 \cos(\omega' t) + C_2 \sin(\omega' t))$ into the differential equation.

(a)

$$\dot{x}(t) = -\gamma e^{-\gamma t} (C_1 \cos(\omega' t) + C_2 \sin(\omega' t)) + e^{-\gamma t} (-C_1 \omega' \sin(\omega' t) + C_2 \omega' \cos(\omega' t))$$

(b)

$$\ddot{x}(t) = e^{-\gamma t} ((\gamma^2 - \omega'^2)(C_1 \cos(\omega' t) + C_2 \sin(\omega' t)) - 2\gamma\omega'(-C_1 \sin(\omega' t) + C_2 \cos(\omega' t)))$$

Substituting $\ddot{x}(t)$, $\dot{x}(t)$, and $x(t)$ back into the equation $m\ddot{x} + b\dot{x} + kx = 0$, we find each term balances, confirming that this is a solution. The underdamped motion is oscillatory with a decaying amplitude due to the factor $e^{-\gamma t}$.

Homework 3: Real and complex scalar fields and dark matter scattering

The interaction Lagrangian in a theory with a real scalar field ϕ , and a complex scalar field ψ is given by:

$$\mathcal{L}_I = y \phi \psi^\dagger \psi.$$

- (1 Pts.) Use the Feynman diagram technique to find the decay amplitude for the field ϕ .
- (3 Pts.) Compute the decay rate Γ for process field $\phi \rightarrow \psi^\dagger \psi$.
- (1 Pts.) For which value of m_ψ is the decay rate maximal?
- (2 Pts.) What is the highest value of m_ψ for which the decay rate is well defined? What is the physical interpretation of this fact?

We can also view this as a simplified model of Dark Matter scattering

- (1 Pts.) Using the Feynman diagram technique, write down the amplitude for $\psi^\dagger(p_1) + \psi(p_2) \rightarrow \phi(q_1) + \phi(q_2)$.
- (4 Pts.) Compute the corresponding total annihilation cross section in the limit that $m_\phi = 0$, and the relative velocity of ψ and ψ^\dagger is low.
- (1 Pts.) What the annihilation rate of ψ and ψ^\dagger , in that case that their number densities are given by $n = n_\psi = n_{\psi^\dagger}$?

SOLUTION:

- a) We have a single diagram for the process $\phi \rightarrow \psi^\dagger \psi$

$$\mathcal{M} = P \text{ --- } \begin{array}{c} \swarrow p \\ \searrow q \end{array} = -iy.$$

- b) From the lecture notes, we have

$$d\Gamma = \frac{1}{2m_\phi} d\Phi_{1 \rightarrow 2} |\mathcal{M}|^2 = \frac{1}{2m_\phi} d\Omega \frac{1}{16\pi^2} \frac{|\vec{p}|}{E_{\text{cm}}} y^2.$$

Setting $E_{\text{cm}} = m_\phi$ and $m_\psi^2 = (m_\phi/2)^2 - |\vec{p}|^2$ we have

$$\Gamma = \frac{y^2}{16\pi m_\phi} \sqrt{1 - \frac{4m_\psi^2}{m_\phi^2}}.$$

- Clearly $m_\psi = 0$ has $\Gamma = y^2/(16\pi m_\phi)$.
- If $m_\phi < 2m_\psi$ the process becomes inaccessible as the two ψ particles cannot be produced.

e) The Feynman diagrams are

$$\mathcal{M} = \begin{array}{c} \psi^\dagger \leftarrow \text{---} \phi \\ \uparrow \\ \psi \rightarrow \text{---} \phi \end{array} + \begin{array}{c} \psi^\dagger \rightarrow \text{---} \phi \\ \downarrow \\ \psi \rightarrow \text{---} \phi \end{array} = (-iy)^2 \left(\frac{1}{(p_1 - q_1)^2 - m_\psi^2} + \frac{1}{(p_1 - q_2)^2 - m_\psi^2} \right)$$

f)

$$\frac{d\sigma}{d\Omega} = \frac{1}{32\pi^2} \frac{|\mathcal{M}|^2 p_f}{4s p_i}$$

The amplitude is angle independent, so the $d\Omega$ integral can be evaluated and yields a factor 4π . Now, in the limit of small v_{rel} , the kinematic variables can be expanded as

$$s \approx 4m_\psi^2 \text{ and } p_i \approx v_{\text{rel}} m_\psi \text{ and } p_f \approx m_\psi.$$

With $(p_1 - q_1)^2 \approx (p_1 - q_2)^2 \approx -m_\psi^2$ the matrix element is also expanded

$$\mathcal{M} \approx \frac{y^2}{m_\psi^2}.$$

Thus we have (including the symmetry factor for identical final state particles)

$$\sigma = \frac{1}{128\pi} \frac{1}{v_{\text{rel}}} \frac{1}{m_\psi^2} \frac{y^4}{m_\psi^4}.$$

g) The annihilation rate is given by

$$\Gamma_{\text{ann}} = \sigma n_\psi v_{\text{rel}} = \frac{n_\psi}{128\pi} \frac{1}{m_\psi^2} \frac{y^4}{m_\psi^4}$$

Homework 4: Renormalisation

Consider the bare Lagrangian

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\phi_0)(\partial^\mu\phi_0) - \frac{1}{2}m_0^2\phi_0^2 - \frac{\lambda_0}{3!}\phi_0^4$$

of a real scalar quantum field theory.

- a) (10 Pts.) Calculate the 1PI contributions to $\langle\Omega|T\{\phi\phi\}|\Omega\rangle$ and $\langle p_3, p_4|S|p_1, p_2\rangle$ for vanishing momenta.
- b) (2 Pts.) Derive the counterterm Lagrangian by replacing the bare parameter $\phi_0 \rightarrow Z_\phi^{1/2}\phi$, $m_0 \rightarrow Z_m m$ and $\lambda_0 \rightarrow Z_\lambda \lambda_0$ in terms of renormalised parameters and separating the counterterm Lagrangian from the renormalised Lagrangian.
- c) (5 Pts.) Express the degree of divergence of a diagram as a function of the number of vertices V and external lines N_e .
- d) (2 Pts.) For which N_e are diagrams finite after all subdivergences are subtracted?
- e) (2 Pts.) Explain if this theory is renormalisable.

SOLUTION: See lecture notes.

- a) cf. eqs. (414) and (418) or (423) and (424)
- b) cf. eq. (428)
- c) cf. eq. (454)
- d) cf. Figure 8
- e) cf. Section 7.4.1

Homework 5: Euler-Lagrange equations for $\phi^3 + \phi^4$

Consider the Lagrangian density $\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_I$, with

$$\mathcal{L}_0 = \frac{1}{2}(\partial_\mu\phi)(\partial^\mu\phi) - \frac{1}{2}m^2\phi^2 \quad \text{and} \quad \mathcal{L}_I = -\frac{\lambda}{3!}\phi^3 - \frac{\lambda}{4!}\phi^4.$$

a) (6 Pts.) Derive the equation of motion from

$$\partial^\mu \left[\frac{\partial \mathcal{L}}{\partial(\partial^\mu\phi)} \right] - \frac{\partial \mathcal{L}}{\partial\phi} = 0.$$

b) (4 Pts.) With the generalized momentum $\pi(x) = (\partial\mathcal{L})/(\partial\dot{\phi})$ write down the Hamiltonian defined by

$$H = \int d^3x [\pi\dot{\phi} - \mathcal{L}].$$

c) (10 Pts.) Using the canonical commutation relations, show that $i\dot{\pi} = [\pi, H]$.

SOLUTION:

a) Using $(\partial_\mu\phi)(\partial^\mu\phi) = \eta_{\rho\sigma}\partial^\rho\phi\partial^\sigma\phi$, we have

$$\frac{\partial}{\partial(\partial^\mu\phi)}(\partial\phi)^2 = 2\partial_\mu\phi.$$

Therefore,

$$\partial^\mu\partial_\mu\phi = -m^2\phi - \frac{1}{2!}\lambda_3\phi^2 - \frac{1}{3!}\lambda_4\phi^3$$

b) The generalised momentum is unchanged from the free case $\pi = \dot{\phi}$, so

$$H = \int d^3x \left[\frac{1}{2}\dot{\phi}^2 + \frac{1}{2}(\nabla\phi)^2 + \frac{1}{2}m^2\phi^2 + \frac{1}{3!}\lambda_3\phi^3 + \frac{1}{4!}\lambda_4\phi^4 \right].$$

c) Using commutation algebra we can write

$$\begin{aligned} [\pi(\vec{x}), \phi(\vec{y})^3] &= \phi(\vec{y})[\pi(\vec{x}), \phi(\vec{y})^2] + [\pi(\vec{x}), \phi(\vec{y})]\phi(\vec{y})^2 = -3i\phi(\vec{y})^2 \delta(\vec{x} - \vec{y}). \\ [\pi(\vec{x}), \phi(\vec{y})^4] &= -4i\phi(\vec{y})^3 \delta(\vec{x} - \vec{y}). \end{aligned}$$

We have already shown that

$$[\pi(\vec{x}), H_0] = i(\nabla^2\phi(\vec{x}) - m^2\phi(\vec{x}))$$

and now can show that

$$[\pi(\vec{x}), H_I] = -i \int \left[\lambda_3 \frac{1}{2} \delta(\vec{x} - \vec{y}) \phi(\vec{y})^2 + \lambda_4 \frac{1}{3!} \phi(\vec{y})^3 \right] d^3y.$$

Therefore,

$$[\pi(\vec{x}), H] = i \left[\nabla^2\phi(\vec{x}) - m^2\phi(\vec{x}) - \frac{1}{2!}\lambda_3\phi(\vec{x})^2 - \frac{1}{3!}\lambda_4\phi(\vec{x})^3 \right] = i\dot{\pi}(\vec{x}).$$

Homework 6: Dirac field (6 Pts.)

The Dirac field may be written

$$\psi(x) = \int \frac{d^3p}{(2\pi)^3 \sqrt{2E_{\vec{p}}}} \sum_{s=1}^2 \left[\psi_p^{(s)}(x) a_s(\vec{p}) + \tilde{\psi}_p^{(s)}(x) b_s^\dagger(\vec{p}) \right],$$

where

$$\begin{aligned} \psi_p^{(s)}(x) &= e^{-ip \cdot x} u_s(p), \\ \tilde{\psi}_p^{(s)}(x) &= e^{ip \cdot x} v_s(p). \end{aligned}$$

With the scalar product $\langle \psi_1 | \psi_2 \rangle = \int d^3x \psi_1^\dagger \psi_2$, show that

$$\langle \psi_p^{(r)} | \psi_q^{(s)} \rangle = \delta_{rs} (2\pi)^3 2E_{\vec{p}} \delta^{(3)}(\vec{p} - \vec{q}).$$

SOLUTION: We start with

$$\langle \psi_p^{(r)} | \psi_q^{(s)} \rangle = \int d^3x \psi_p^{(r)\dagger} \psi_q^{(s)} = e^{i(p^0 - q^0)t} \int d^3x e^{-i(\vec{p} - \vec{q}) \cdot \vec{x}} u_r(p)^\dagger u_s(q).$$

We now use the solution for u_r from the lecture notes

$$u_r(p) = \frac{\gamma \cdot p + m}{\sqrt{2m(m + E_{\vec{p}})}} u_r(0)$$

as well as

$$\int d^3x e^{-i(\vec{p} - \vec{q}) \cdot \vec{x}} = (2\pi)^3 \delta^{(3)}(\vec{p} - \vec{q})$$

to write

$$\langle \psi_p^{(r)} | \psi_q^{(s)} \rangle = e^{i(p^0 - q^0)t} (2\pi)^3 \delta^{(3)}(\vec{p} - \vec{q}) u_r(0)^\dagger \frac{(\gamma \cdot p + m)^\dagger}{\sqrt{2m(m + E_{\vec{p}})}} \frac{\gamma \cdot q + m}{\sqrt{2m(m + E_{\vec{q}})}} u_s(0).$$

Since the δ function forces $\vec{p} = \vec{q}$, we can simplify this

$$\langle \psi_p^{(r)} | \psi_q^{(s)} \rangle = (2\pi)^3 \delta^{(3)}(\vec{p} - \vec{q}) u_r(0)^\dagger \frac{(\gamma^\dagger \cdot p + m)(\gamma \cdot p + m)}{2m(m + E_{\vec{p}})} u_s(0).$$

Through explicit calculation we have for the numerator

$$(\gamma^\dagger \cdot p + m)(\gamma \cdot p + m) = \begin{pmatrix} 2E_{\vec{p}}(E_{\vec{p}} + m) & 0 & -2E_{\vec{p}}p_z & -2E_{\vec{p}}(p_x - ip_y) \\ 0 & 2E_{\vec{p}}(E_{\vec{p}} + m) & -2E_{\vec{p}}(p_x + ip_y) & 2E_{\vec{p}}p_z \\ -2E_{\vec{p}}p_z & -2E_{\vec{p}}(p_x - ip_y) & 2E_{\vec{p}}(E_{\vec{p}} - m) & 0 \\ -2E_{\vec{p}}(p_x + ip_y) & 2E_{\vec{p}}p_z & 0 & 2E_{\vec{p}}(E_{\vec{p}} - m) \end{pmatrix}$$

Considering $u_1 = (\sqrt{2m}, 0, 0, 0)$ and $u_2 = (0, \sqrt{2m}, 0, 0)$, we have four cases to consider. These correspond to the 1,1, 1,2, 2,1 and 2,2 elements of the matrix above. Had we considered $v_1 = (0, 0, \sqrt{2m}, 0)$ and $v_2 = (0, 0, 0, \sqrt{2m})$ we would work with the 3,3, 3,4, 4,3 and 4,4 matrix elements. These can be written as

$$\begin{aligned} u_r(0)^\dagger (\dots) u_s(0) &= \delta_{rs} 4m E_{\vec{p}} (E_{\vec{p}} + m), \\ v_r(0)^\dagger (\dots) v_s(0) &= \delta_{rs} 4m E_{\vec{p}} (E_{\vec{p}} - m). \end{aligned}$$

Therefore,

$$\langle \psi_p^{(r)} | \psi_q^{(s)} \rangle = (2\pi)^3 \delta^{(3)}(\vec{p} - \vec{q}) \delta_{rs} 2E_{\vec{p}}.$$

Homework 7: Wick theorem

Use the Wick theorem to calculate the symmetry factors of the diagrams in (190), (193), (210), (211), (212), (213), (214) of the lecture notes.

SOLUTION: See lecture notes.