MATH425 – Quantum Field Theory

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Abstract

Quantum Field Theory is the unification of special relativity with quantum mechanics. First attempts at this failed because physicists tried to keep the number of particles constant. For energies much smaller than the rest mass, i.e. $E \ll mc^2$, this is a reasonable approximation. However, once we thinking about high-energy processes we can create new particles. Once this idea was accepted, the development of Quantum Field Theory was still somewhat rocky. This changed when in 1948 Julian Schwinger calculated the anomalous magnetic moment of the electron. The following decades are followed are the story of the success of Quantum Field Theory. The Quantum Field Theory prediction of the anomalous magnetic moment of the electron is to this day the most precise test of a theory in all of science:

$$a_e^{\text{th}} = 0.001\,159\,652\,181\,643(764)$$
$$a_e^{\text{exp}} = 0.001\,159\,652\,180\,59(13)\,.$$

I will not be able to show you how this is calculated. But I hope that, over the coming weeks and months, I can give you an idea of the underpinnings of this calculation.

We will begin by studying a relativistic classical scalar field with the Klein-Gordon equation. Next, we will quantise this field, first without interactions and then with using perturbation theory. In the end, we will study decay and scattering as well as higher-order corrections.

For an up-to-date copy of these notes, see `https://math425.yannickulrich.com/notes`. Please report any mistakes at `https://gitlab.com/yannickulrich/qft/-/issues`.

Contents

1 Classical scalar field

In quantum mechanics (QM) we only considered non-relativistic single particles. By then special relativity was already well-known and many physicists tried to find a combined theory of special relativity and QM, similar to how we are trying to find a combined theory with general relativity today. The first attempt at this was the Klein-Gordon (KG) equation which, in essence, was very similar to the Schrödinger equation. Recall how we identify operators in QM

$$p \to -i\hbar\partial_x \quad \text{and} \quad E \to i\hbar\partial_t\,. \tag{1}$$

The Schrödinger equation is just the definition of the non-relativistic energy $E = T + V$

$$\begin{aligned} E = T + V = \frac{p^2}{2m} + V\,, \\ i\hbar\partial_t\Psi = \Big[-\frac{\hbar^2}{2m}\partial_x^2 + V\Big]\Psi\,. \end{aligned} \tag{2}$$

It is now natural to try and extend this to the relativistic case $E^2 - c^2p^2 = c^4m^2$ (which of course for $p = 0$ is just $E = mc^2$)

$$\hbar^2\Big(-\frac{1}{c^2}\partial_t^2 + \partial_x^2\Big)\Psi = m^2c^2\Psi\,. \tag{3}$$

This is the KG equation. Unfortunately, this equation is, as it stands, inconsistent and Schrödinger himself discarded it immediately, choosing to instead expand the relativistic mass-energy relation which lead him to (2).

The solution to this problem will be that we need to consider the number of particles free rather than fixed to one. Recall how in perturbation theory we had to sum over all possible states rather than just the lowest energy one. Here we have much the same.

Before we can do this, we need to briefly revisit some aspects of classical physics. We will spend the rest of this chapter reviewing relativity and Lagrangian mechanics and classical field theory.

1.1 Relativity

At its heart, physics is the study of the symmetries of nature and their consequences. By studying how systems change or more importantly do not change under transformation, we can identify important properties of the system (cf. Noether theorem). Arguably one of the most important symmetries there is, is Lorentz symmetry, i.e. invariance under spacetime rotations in special relativity. These consists of rotation in 3D space and boosts. The following is merely a brief summary of what is required for this course and should not be viewed as complete.

1.1.1 Rotation

A rotation in 3D can be expressed using a 3×3 matrix R such that

$$\begin{aligned} \vec{x} \to \vec{x}\,' = R\vec{x}\,, \\ x_i \to x_i' = \sum_{j=1}^{3} R_{ij}x_j = R_{ij}x_j\,. \end{aligned} \tag{4}$$

Here we have started using Einstein sum conventions where a sum is implicit over indices that appear exactly twice. Since a rotation of the entire system is supposed to not change the lengths of vectors or the angles between them, let us consider what happens to a scalar product between two vectors $\vec{x}$ and $\vec{y}$

$$\vec{x}\cdot\vec{y} = \sum_{i=1}^{3} x_iy_i = x_iy_i \to x_i'y_i' = R_{ij}x_jR_{ik}x_k\,. \tag{5}$$

The only way for this to hold is if

$$R_{ij}R_{ik} = R_{ji}(R^T)_{ik} = \delta_{jk} \quad \text{or in other words} \qquad RR^T = 1\,, \tag{6}$$

with the Kronecker delta δ_{jk}. This means that R has to be an orthogonal matrix. Since we usually want to disallow reflections, we also require $\det R = 1$ to arrive at the symmetry group

$$\mathrm{SO}(3) = \left\{ R \;\middle|\; RR^T = 1 \text{ and } \det R = 1 \right\} . \tag{7}$$

1.1.2 Minkowski space

What is the equivalent of distances and rotations in special-relativity, i.e. what do we require to remain invariant under transformation? Measurements tell us that the 'distance' in spacetime is defined as

$$s^2 = c^2 t^2 - \vec{x}^2 \,, \tag{8}$$

which remains invariant compared to the previous $s^2 = \vec{x}^2$. This is very similar to the rotations except for the extra sign. In 3D we have worked in Euclidean space while we now need to work in Minkowskian space. Similarly, the metric s is now called the Minkowski metric[1]. For simplicity, let us collect the time component into the vector x to write the vector

$$x^\mu = (x^0, x^1, x^2, x^3) = (ct, \vec{x}) \,, \tag{9}$$

which we need to distinguish from the covector which has lower indices

$$x_\mu = (x^0, -x^1, -x^2, -x^3) = (ct, -\vec{x}) \,, \tag{10}$$

to properly account for the metric. In sum convention, we may only contract upper with lower indices, i.e.

$$x^\mu x_\mu = c^2 t^2 - \vec{x}^2 = s^2 \,, \tag{11}$$

is valid while $x^\mu y^\mu$ is not. It would therefore be helpful to have a way to raise or lower indices which is done using the metric itself

$$\eta_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} . \tag{12}$$

Signs of the metric
Note that there are two competing conventions for the signs in η. In this lecture (and the broader particle physics community) we will use the 'mostly minus' conventions, sometimes called west coast metric. Alternatively, people use a metric which has the minus sign in the time component which is more common in string theory and cosmology. This is often referred to as the 'mostly plus' or east coast convention. The west coast convention has the nice property that the momentum of a massive particle squares to its mass, i.e. $p^\mu p_\mu = m^2$ rather than $p^\mu p_\mu = -m^2$. When reading other resources, please make sure you understand the metric the author uses to avoid making sign mistakes!

η takes a role not dissimilar from the Kronecker delta in Euclidean space. We can now write

$$x_\mu = \eta_{\mu\nu} x^\nu \quad \text{and} \quad x^\mu = \eta^{\mu\nu} x_\nu \,, \tag{13}$$

where we have defined the new $\eta^{\mu\nu}$. Luckily, its matrix representation is the same since

$$x^\mu = \eta^{\mu\nu} \eta_{\nu\rho} x^\rho \quad \text{and therefore} \quad \eta^{\mu\nu} \eta_{\nu\rho} = \delta^\mu{}_\rho \,. \tag{14}$$

[1] Formally, the Minkowski metric is not a metric in the mathematical sense because s^2 can be negative or zero without $x = 0$.

> **Exercise**: What is $\eta_{\mu\nu}\eta^{\nu\mu}$?

Please note that for a general tensor, i.e. an object with multiple indices, the order of indices matters. You can easily convince yourself that it does not for the Kronecker delta $\delta^{\mu}{}_{\rho}$ but you should be careful. Raising and lowering indices works also for tensors, e.g.

$$w^{\mu\nu} = \eta^{\nu\rho} w^{\mu}{}_{\rho} = \eta^{\mu\sigma}\eta^{\nu\rho} w_{\sigma\rho} \,. \tag{15}$$

It is very important to remember that $w^{\mu\nu}$, $w^{\mu}{}_{\rho}$, $w_{\sigma\rho}$ and even $w_{\mu}{}^{\rho}$ are all different objects!

Another important object to consider is the derivative operator which also exists as a vector and a covector

$$\partial^{\mu} = \frac{\partial}{\partial x_{\mu}} = \left(\frac{1}{c}\partial_t, -\vec{\nabla}\right) , \tag{16}$$

$$\partial_{\mu} = \frac{\partial}{\partial x^{\mu}} = \left(\frac{1}{c}\partial_t, +\vec{\nabla}\right) . \tag{17}$$

The derivative of x itself works as expected

$$\partial_{\mu} x^{\nu} = \delta_{\mu}^{\ \nu} \quad \text{and} \quad \partial^{\mu} x_{\nu} = \delta^{\mu}_{\ \nu} \,. \tag{18}$$

One can easily show that the momentum

$$p^{\mu} = \Big(E/c, \vec{p}\Big) \tag{19}$$

is a vector which allows the identification (1).

1.1.3 Lorentz transformation

We now have the tools to actually study Lorentz transformations. We begin by defining

- the scalar product in Minkowski space which works the same for vectors and covectors

$$x \cdot y = x^{\mu} y_{\mu} = x_{\mu} y^{\mu} = \eta_{\mu\nu} x^{\mu} y^{\nu} = \eta^{\mu\nu} x_{\mu} y_{\nu} \,. \tag{20}$$

- the Lorentz transformation as a linear transformation

$$x^{\mu} \to (x')^{\mu} = \Lambda^{\mu}{}_{\nu} x^{\nu} \,. \tag{21}$$

> **Exercise**: Find an explicit form of Λ.

We want this transformation to preserve scalar products, i.e. $x' \cdot y' = x \cdot y$

$$x' \cdot y' = \eta_{\mu\nu} (x')^{\mu} (y')^{\nu} = \eta_{\mu\nu} \Lambda^{\mu}{}_{\rho} x^{\rho} \Lambda^{\nu}{}_{\sigma} y^{\sigma} \stackrel{!}{=} \eta_{\sigma\rho} x^{\rho} y^{\sigma} \,. \tag{22}$$

This allows us to specify the full Lorentz group

$$\mathrm{O}(1,3) = \Big\{ \Lambda \ \Big| \ \eta_{\mu\nu} \Lambda^{\mu}{}_{\rho} \Lambda^{\nu}{}_{\sigma} = \eta_{\sigma\rho} \Big\} \,. \tag{23}$$

This relation is very similar to the orthogonal group O(3) and hence has a similar name. We use the arguments $1,3$ to indicate that there is one time dimension and three spacial dimensions that have opposite signs. Similarly to how we wanted rotations to not include reflections, we can define subgroups

- orthochronous $\mathrm{O}^{+}(1,3)$ which preserves the direction of time by requiring that $\Lambda^{0}{}_{0} \geq 1$.
- proper $\mathrm{SO}(1,3)$ which preserves orientation by requiring that $\det \Lambda = +1$.

- improper which flips orientation, i.e. $\det \Lambda = -1$
- non-orthochronous $\mathrm{O}^-(1,3)$ which flips the direction of time by requiring that $\Lambda^0{}_0 \leq 1$.

When we talk about the Lorentz group, we often refer to the proper orthochronous group $\mathrm{SO}^+(1,3)$.

Exercise: Proof that $\mathrm{SO}^+(1,3)$ is a group.

Exercise: Proof that $p^2 = p \cdot p = p^\mu p^\nu \eta_{\mu\nu}$ is invariant under Lorentz transformation. What does this mean for the mass-energy relation $E^2/c - \vec{p}^2 = m^2c^2$?

When we combine the Lorentz group with invariance under shifts, we obtain the largest group of spacetime symmetry, the Poincaré group.

Since the Lorentz group is connect, we can obtain every element of $\mathrm{SO}^+(1,3)$ by concatenating infinitesimal Lorentz transforms starting from the identity transform δ, i.e.

$$\Lambda^\mu{}_\nu = \delta^\mu{}_\nu + \omega^\mu{}_\nu + \mathcal{O}(\omega^2)\,. \tag{24}$$

By substituting this into the condition for the Lorentz group (23), we find

$$\begin{aligned} \eta_{\sigma\rho} &= \eta_{\mu\nu}\Lambda^\mu{}_\rho\Lambda^\nu{}_\sigma = \eta_{\mu\nu}\left(\delta^\mu{}_\rho + \omega^\mu{}_\rho\right)\left(\delta^\nu{}_\sigma + \omega^\nu{}_\sigma\right) + \mathcal{O}(\omega^2) \\ &= \eta_{\rho\sigma} + \eta_{\rho\nu}\omega^\nu{}_\sigma + \eta_{\mu\sigma}\omega^\mu{}_\rho + \mathcal{O}(\omega^2) = \eta_{\rho\sigma} + \omega_{\rho\sigma} + \omega_{\sigma\rho} + \mathcal{O}(\omega^2)\,. \end{aligned} \tag{25}$$

In other words, ω is antisymmetric

$$\omega_{\rho\sigma} = -\omega_{\sigma\rho}\,, \tag{26}$$

which is why it is so important to keep the order of indices correct.

Aside from (21), it is also useful to know how derivatives transform

$$\frac{\partial}{\partial x^\mu} = \frac{\partial (x')^\nu}{\partial x^\mu}\frac{\partial}{\partial (x')^\nu} = \Lambda^\nu{}_\mu \frac{\partial}{\partial (x')^\nu}\,. \tag{27}$$

1.1.4 Transformation of the KG field

If we want $\mathrm{SO}^+(1,3)$ to be a symmetry of nature, our theories need to invariant under transformation. To ensure this, we need to study how for example the solution $\Psi(x)$ of the KG equation (3) transforms. $\Psi(x)$ maps every point x in spacetime to a (complex) number $\Psi(x)$. If x is for example the point where $\Psi(x)$ is maximal, this property needs to be retained even after transformation, i.e.

$$\begin{aligned} x &\to x' = \Lambda x\,, \\ \Psi(x) &\to \Psi'(x') = \Psi'(\Lambda x) \stackrel{!}{=} \Psi(x)\,. \end{aligned} \tag{28}$$

The way to ensure this, is to require

$$\Psi(x) \to \Psi'(x) = \Psi(\Lambda^{-1}x)\,. \tag{29}$$

For the KG equation we need

$$\eta^{\mu\nu}\partial_\mu\partial_\nu\Psi(x) \to \underbrace{\eta^{\mu\nu}(\Lambda^{-1})^\rho{}_\mu(\Lambda^{-1})^\sigma{}_\nu}_{\eta^{\rho\sigma}}(\partial_\rho\partial_\sigma\Psi)(\Lambda^{-1}x) = (\eta^{\rho\sigma}\partial_\rho\partial_\sigma\Psi)(\Lambda^{-1}x)\,, \tag{30}$$

where we have used again the defining properties of Λ. With this it is obvious that the KG equation (3) (now rewritten in more compact notation)

$$(\hbar^2\partial_\mu\partial^\mu + m^2c^2)\Psi = 0 \tag{31}$$

is invariant

$$(\hbar^2\partial_\mu\partial^\mu + m^2c^2)\Psi(x) \to (\hbar^2\partial_\mu\partial^\mu + m^2c^2)\Psi(\Lambda^{-1}x) \tag{32}$$

1.2 A brief digression on units

In the above discussion we often had to write $\hbar$ and c. To avoid doing this, it is common to choose a unit system where $\hbar = c = 1$ and we only have a single unit (such as GeV). This is perfectly permissible as we will always know how to convert back to SI units by multiplying with the correct powers of $\hbar$ and c. With the 2019 revision of the SI system, both c and $\hbar$ have definite values without uncertainties as they are used to define the meter and kilogram respectively. A helpful value to remember is $(\hbar c) = 197.3\,\mathrm{MeV}\cdot\mathrm{fm}$.

Exercise: Convert the following values back to SI units:

- the total cross section for pp collisions at the LHC is $\sigma \approx 250\,\mathrm{GeV}^{-2}$. How much is this in fb^2?
- the muon lifetime is $\tau \approx 3.3\times 10^{18}\,\mathrm{GeV}^{-1}$. How much is this in μs?
- the electron mass is $m \approx 0.511\,\mathrm{MeV}$. How much is this in kg?

1.3 Lagrangian mechanics

Let us briefly review a classical system with n degrees of freedom such as a collection of $n/3$ particles that can all move independently of each other. This system is completely described by n generalised coordinates $q_1, ..., q_n$ and n generalised velocities $\dot{q}_1, ..., \dot{q}_n$. To find these, we need the Lagrangian L. For a system of particles, this is

$$L(\{q_i\},\{\dot{q}_i\},t) = T - V = \frac{1}{2}\sum_{k=1}^{n} m_i \dot{q}_i^2 - V(\{q_i\},\{\dot{q}_i\})\,. \tag{33}$$

To derive the equations of motions (EoMs) for this system, we define the *action* S functional

$$S[\{q_i(t)\}] = \int_{t_0}^{t_1} \mathrm{d}t\; L(\{q_i\},\{\dot{q}_i\},t) \tag{34}$$

and use the variational principle, i.e. we require the action is extremal[2]

$$\delta S = 0\,. \tag{35}$$

From this you can easily derive the Euler-Lagrange equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = 0\,. \tag{36}$$

Derivation of the Euler-Lagrange for $n = 1$
In the $n = 1$ case we have only one q so that the action only depends on the function $q(t)$. Fixing the boundary conditions $q(t_0) = q_0$ and $q(t_1) = q_1$, we vary the path by a small $\epsilon\,\omega(t)$ with $\omega(t_0) = \omega(t_1) = 0$. Then,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}\epsilon}S[q+\epsilon\,\omega] &= \int_{t_0}^{t_1}\mathrm{d}t\;\frac{\mathrm{d}}{\mathrm{d}\epsilon}L\big(q(t)+\epsilon\,\omega(t),\dot{q}(t)+\epsilon\,\dot{\omega}(t),t\big)\\ &= \int_{t_0}^{t_1}\mathrm{d}t\;\left[\omega(t)\frac{\partial L}{\partial q}\big(q(t)+\epsilon\,\omega(t),\dot{q}(t)+\epsilon\,\dot{\omega}(t),t\big) + \dot{\omega}(t)\frac{\partial L}{\partial \dot{q}}\big(q(t)+\epsilon\,\omega(t),\dot{q}(t)+\epsilon\,\dot{\omega}(t),t\big)\right],\end{aligned} \tag{37}$$

[2] This is sometimes referred to as the principle of least action. However, the action does not need to minimised (even if it often is) as long as it is extremal

since t does not depend on ϵ. For $\epsilon = 0$, we have an extremal value

$$\frac{\mathrm{d}S[q]}{\mathrm{d}\epsilon} = \int_{t_0}^{t_1} \mathrm{d}t \left[\omega(t)\frac{\partial L}{\partial q}\big(q(t),\dot{q}(t),t\big) + \dot{\omega}(t)\frac{\partial L}{\partial \dot{q}}\big(q(t),\dot{q}(t),t\big)\right] . \tag{38}$$

Applying integration-by-parts on the second term with $\omega(t_0) = \omega(t_1) = 0$ such hat the boundary conditions vanish

$$\frac{\mathrm{d}S[q]}{\mathrm{d}\epsilon} = \int_{t_0}^{t_1} \mathrm{d}t\ \omega(t)\left[\frac{\partial L}{\partial q}\big(q(t),\dot{q}(t),t\big) - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}}\big(q(t),\dot{q}(t),t\big)\right] . \tag{39}$$

Since ω is an arbitrary smooth function, the fundamental lemma of the calculus of variations requires that the bracket vanishes.

Because of its prevalence in mechanics, we define the conjugate momentum to q_i

$$\pi_i = \frac{\partial L}{\partial \dot{q}_i} \tag{40}$$

Next, we can define the Hamiltonian

$$H = \sum_{j=1}^{n} \dot{q}_j \pi_j - L(\{q_i\},\{\dot{q}_i\},t) , \tag{41}$$

which in the case of (33) is just the total energy of the system

$$H = \sum_{j=1}^{n} \dot{q}_j \underbrace{\frac{\partial L}{\partial \dot{q}_i}}_{\dot{q}_j m_j} - L = T + V . \tag{42}$$

We can now wonder when energy is conversed by calculating the total derivative of H w.r.t. t

$$\frac{\mathrm{d}H}{\mathrm{d}t} = \underbrace{\ddot{q}_j\frac{\partial L}{\partial \dot{q}_j} + \dot{q}_j\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_j}}_{\mathrm{d}(q_j\pi_j)/\mathrm{d}t} \underbrace{- \frac{\partial L}{\partial q_j}\dot{q}_j - \frac{\partial L}{\partial \dot{q}_j}\ddot{q}_j - \frac{\partial L}{\partial t}}_{-\mathrm{d}L/\mathrm{d}t} , \tag{43}$$

where we have left the sum over j implicit. Using (36), we can cancel the remaining terms and are left with

$$\frac{\mathrm{d}H}{\mathrm{d}t} = -\frac{\partial L}{\partial t} . \tag{44}$$

In other words, if the Lagrangian L does not explicitly depend on t (for example through a time-dependent potential $V(t)$), energy is conserved. This is the first example of what is often called a Noether current or Noether charge, that is a conserved quantity that arises because the Lagrangian has a certain symmetry such as time-invariance.

Noether Theorem
In 1918, the German mathematician Emmy Noether proved that if the Lagrangian L is invariant under small perturbations of the time variable and the coordinates q_i, there exists a conserved quantity for each of the n coordinates. To quantify this, let T be the generator of time evolution and Q_i the generator of the symmetry

$$t \to t + \delta t = t + \epsilon T , \tag{45}$$

$$q_i \to q_i + \delta q_i = q_i + \epsilon Q_i . \tag{46}$$

If L is invariant under the this transformation,

$$\mathbb{O} = H\,T - \pi_i Q_i \tag{47}$$

is conserved. Examples include

- $T = 1$, $Q_i = 0$: energy is conserved if the potential is not time-dependent.
- $T = 0$, $Q_i = 1$: linear momentum is conserved if the potential is shift-invariant.
- $T = 0$, for each particle r $\vec{Q}_r = \vec{n} \times \vec{q}_r$ with some vector $\vec{n}$: angular momentum along the axis $\vec{n}$ is conserved if the Lagrangian is spherical symmetric.
- If the L is invariant under Lorentz boost, the centre-of-mass system moves with constant velocity.

Additionally to Lagrangian mechanics, sometimes it is helpful to consider the Hamiltonian EoMs

$$\dot{q}_j = \frac{\partial H}{\partial \pi_j} \quad \text{and} \quad \dot{\pi}_j = -\frac{\partial H}{\partial q_j}\,. \tag{48}$$

Exercise: Derive these using the Euler-Lagrange equation.

1.4 Classical field theory

Before we can study quantum field theorys (QFTs), let us review classical field theories. A classical field $\phi(\vec{x}, t)$ is a function that can take a value for each point in space and time. From a Lagrangian point of view, this means that we one degree of freedom for each point, requiring us to use integrals rather than finite sums. Since we also want to consider relativity, we further want to avoid talking about space differently from time. Therefore, the Lagrangian functional L is now less useful and we instead use the Lagrangian density $\mathcal{L}$ which confusingly is also often called Lagrangian. The generalised coordinate now becomes the field $\phi(x)$ and the generalised velocity the derivative of the field $\partial^\mu\phi$. The action functional is still defined the same way

$$S[\phi] = \int \mathrm{d}t\ L[\phi] = \int \mathrm{d}t \int \mathrm{d}^3x\ \mathcal{L}[\phi, \partial^\mu\phi] = \int \mathrm{d}^4x\ \mathcal{L}[\phi, \partial^\mu\phi]\,. \tag{49}$$

Similarly to the $n = 1$ case above, we can derive the Euler-Lagrange equation by requiring $\delta S = 0$

$$\frac{\partial}{\partial x_\mu}\frac{\partial \mathcal{L}}{\partial(\partial^\mu\phi)} - \frac{\partial \mathcal{L}}{\partial \phi} = \frac{\partial}{\partial x_\mu}\pi_\mu - \frac{\partial \mathcal{L}}{\partial \phi} = 0\,. \tag{50}$$

where we have once again defined the conjugate momentum π.

Exercise: Do this by adding a small variation $\phi \to \phi + \delta\phi$ and use Gauss's law to remove the surface terms.

Now let us consider the KG as defining a field ϕ instead of a wavefunction Ψ and derive the EoM (31). We first require the Lagrangian $\mathcal{L}$. While we could write down a field theory that depends on ϕ and $\partial^\mu\phi$ in any way we like, we usually want $\mathcal{L}$ to be polynomial in the fields and derivatives. There are plenty of examples where this is not the case though (such as the Sine-Gordon theory or Higgs Effective Theory). We do require $\mathcal{L}$ to be a Lorentz scalar though, meaning that it cannot have open indices, and that $\mathcal{L}$ has units of GeV^4 so that the action S is dimensionless. The first conditions implies that $\mathcal{L}$ can only be a function of $(\partial_\mu\phi)(\partial^\mu\phi)$ as there are no other vectors to contract with. It turns out that

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\phi)(\partial^\mu\phi) - \frac{1}{2}m^2\phi^2\,. \tag{51}$$

Note that the normalisation of 1/2 in front of the kinetic term does not matter classically as it does not impact the Euler-Lagrange equations. This changes once we start studying quantum fields so we will keep it already canonically normalised now. Let us that this reduces to the KG equation by computing

$$\frac{\partial \mathcal{L}}{\partial(\partial^\mu \phi)} = \frac{1}{2}\frac{\partial}{\partial(\partial^\mu \phi)}\Big[\eta_{\alpha\beta}(\partial^\alpha\phi)(\partial^\beta\phi)\Big] = \frac{1}{2}\eta_{\alpha\beta}\Big[(\partial^\alpha\phi)\delta^\beta{}_\mu + \delta^\alpha{}_\mu(\partial^\beta\phi)\Big] = \partial_\mu\phi\,. \tag{52}$$

Therefore, we find for (50)

$$\frac{\partial}{\partial x_\mu}\partial_\mu\phi - \frac{1}{2}\frac{\partial(-m^2\phi^2)}{\partial\phi} = (\partial^\mu\partial_\mu + m^2)\phi = 0\,, \tag{53}$$

which is just (31).

For the Hamiltonian formulation, we need the conjugate momentum which is written through the time derivative

$$\pi(x) = \frac{\partial\mathcal{L}}{\partial\dot\phi} = \dot\phi(x)\,, \tag{54}$$

which, like ϕ itself, is a scalar field in that it assigns a scalar value to every point in spacetime.

When we considered the case where n is finite, our next subject was to show that total energy is conserved. We can do something very similar here by defining the Hamiltonian density $\mathcal{H}$

$$\mathcal{H} = \dot\phi\pi - \mathcal{L} = \frac{1}{2}\pi^2 + \frac{1}{2}(\nabla\phi)^2 + \frac{1}{2}m^2\phi^2\,. \tag{55}$$

but it is actually helpful to think broader. Once we start considering relativity, energy and momentum become frame-dependent and it would be nice to have a covariant description that works with four-vectors rather than focussing on energy and momentum separately. Such an object is the energy-momentum tensor

$$T^{\mu\nu} = (\partial^\mu\phi)\frac{\partial\mathcal{L}}{\partial(\partial_\nu\phi)} - \mathcal{L}\eta^{\mu\nu}\,, \tag{56}$$

which considers energy and momentum densities but also their fluxes as well as pressure and stress. $T^{\mu\nu}$ is a symmetric tensor which is trivial to see in our specific example,

$$\begin{aligned} T^{\mu\nu} &= (\partial^\mu\phi)(\partial^\nu\phi) - \frac{1}{2}(\partial_\sigma\phi)(\partial_\rho\phi)\eta^{\rho\sigma}\eta^{\mu\nu} + \frac{1}{2}m^2\phi^2\eta^{\mu\nu} \\ &= (\eta^{\rho\mu}\eta^{\sigma\nu} - \frac{1}{2}\eta^{\rho\sigma}\eta^{\mu\nu})(\partial_\sigma\phi)(\partial_\rho\phi) + \frac{1}{2}m^2\phi^2\eta^{\mu\nu}\,. \end{aligned} \tag{57}$$

This is very similar to $\mathcal{H}$ and in fact $T^{00} = \mathcal{H}$ is the energy density.

Since we have four translation symmetries (one time and three spatial ones), we expect four conserved Noether currents

$$\begin{aligned} \partial_\nu T^{\mu\nu} &= (\partial_\nu\partial^\mu\phi)\frac{\partial\mathcal{L}}{\partial(\partial_\nu\phi)} + (\partial^\mu\phi)\left(\partial_\nu\frac{\partial\mathcal{L}}{\partial(\partial_\nu\phi)}\right) - \underbrace{\left(\frac{\partial\mathcal{L}}{\partial\phi}\partial_\nu\phi + \frac{\partial\mathcal{L}}{\partial(\partial_\rho\phi)}\partial_\nu(\partial_\rho\phi)\right)}_{\partial_\nu\mathcal{L}}\eta^{\mu\nu} \\ &= (\partial_\nu\partial^\mu\phi)\frac{\partial\mathcal{L}}{\partial(\partial_\nu\phi)} + (\partial^\mu\phi)\left(\partial_\nu\frac{\partial\mathcal{L}}{\partial(\partial_\nu\phi)}\right) - \left(\partial^\sigma\frac{\partial\mathcal{L}}{\partial(\partial^\sigma\phi)}\partial_\nu\phi + \frac{\partial\mathcal{L}}{\partial(\partial_\rho\phi)}\partial_\nu(\partial_\rho\phi)\right)\eta^{\mu\nu} = 0\,. \end{aligned} \tag{58}$$

Here we have used the chain rule to expand $\partial_\nu\mathcal{L}$ and (50) for the first term in the $\eta^{\mu\nu}$ bracket. We have now shown that the energy-momentum tensor is a conserved quantity in a more differential sense.

Let us pause for a moment to understand what this means by considering the $\mu = 0$ case, i.e. $j^\nu = T^{0\nu}$ for which we still have the same conservation law

$$0 = \partial_\nu j^\nu = \frac{\partial j^0}{\partial t} + \vec\nabla\cdot\vec{j}\,. \tag{59}$$

By integrating this over a small region of space V, we find using Gauss's divergence theorem for the second term

$$\frac{\partial}{\partial t}\int_V \mathrm{d}^3x\, j^0 = -\oint_{\partial V} \mathrm{d}\vec{S}\cdot\vec{j}\,. \tag{60}$$

Since we have already identified j^0 as the energy density, this implies that the change in energy in the volume is equal to the flux of energy out of this volume. We will often encounter conservation laws like this, from energy or momentum like here to probability densities in wavefunctions or electric charge in electrodynamics.

Before concluding this chapter, let us derive the general solution of the KG equation. Since this is a wave equation, we begin by making an ansatz in terms of plane wave solutions

$$\phi(x) = A\mathrm{e}^{-ik\cdot x} + B\mathrm{e}^{+ik\cdot x}\,, \tag{61}$$

where k^μ is a four vector indicating the direction of travel of our plane wave. Since we have considered only the real KG where $\phi = \phi^*$, we also need a real solution, i.e. $B = A^*$. We can constrain k by substituting the ansatz into the KG equation

$$\partial^2\phi + m^2\phi = -k^2\phi + m^2\phi = 0\,. \tag{62}$$

Therefore, $k^2 = m^2$ which means that the momentum of the plane wave needs to be on its so-called mass-shell. Of course, the general solution is a linear combination of plane waves

$$\phi(x) = \int \frac{\mathrm{d}^3k}{(2\pi)^3}\frac{1}{\sqrt{2E_{\vec{k}}}}\Big[a_{\vec{k}}\mathrm{e}^{-ik\cdot x} + a^*_{\vec{k}}\mathrm{e}^{ik\cdot x}\Big] \tag{63}$$

where $k^2 = (k^0)^2 - \vec{k}^2 = E_{\vec{k}}^2 - \vec{k}^2 = m^2$. The factor $1/\sqrt{2E}$ does not really matter at this point but it will become convenient later.

Exercise: Repeat the above discussion for the complex KG field where ϕ and ϕ^* are independent degrees of freedom. Start with the Lagrangian

$$\mathcal{L} = (\partial_\mu\phi)(\partial^\mu\phi^*) - m^2\phi\phi^*\,. \tag{64}$$

In a QFT context, ϕ^* would be the antiparticle to ϕ's particle.

2 Free quantised scalar field

We are now almost ready to quantise the scalar field ϕ, that is, find a QFT that follows the KG field as specified by the Lagrangian (51). In this section we will consider this field to be a free field, i.e. one that has no interactions, not even with itself. This is not a very realistic description of nature but it is a very important first step. Our description of realistic interacting fields will largely be based on free fields and we will consider the interaction as a small perturbation.

Our discussion is analogous to going from a finite-dimensional Lagrangian problem, where we had n degrees of freedom that each took a numeric value, to a classical field theory, where we assigned each point in spacetime a value $\phi(x)$. When describing such a system using QM, we defined operators for the position and momentum of each of these particles. Similarly, a quantum field is assigns an operator to every point in spacetime. Before we can actually write down these fields we need to do a bit more revising.

2.1 Heisenberg and Schrödinger pictures

Most discussions of QM describe the wavefunction as dynamic and time dependent and the operators acting on them as static. This is called the Schrödinger picture which describes wavefunctions as manifestly time dependent through the Schrödinger equation (2)

$$i\frac{\partial}{\partial t}\Psi_S(x,t) = \underbrace{\Big[-\frac{1}{2m}\frac{\partial^2}{\partial x^2}+V\Big]}_{\hat{H}_S}\Psi_S(x,t)\,. \tag{2}$$

The Hamiton operator $\hat{H}_S$ itself does not depend on time.[3] Starting from some initial state $|\Psi_S(0)\rangle$, the time evolution of the state is described by

$$|\Psi_S(t)\rangle = \hat{U}(t)|\Psi_S(0)\rangle\,. \tag{65}$$

The operator $\hat{U}(t)$ fulfils the differential equation

$$i\frac{\partial \hat{U}}{\partial t} = \hat{H}_S\hat{U}\,. \tag{66}$$

Please keep in mind that this is a differential equation for *an operator* rather than a function. To ensure that $\langle\Psi|\Psi\rangle$ remains unchanged, we need the operator $\hat{U}$ to be unitary, i.e. $\hat{U}^{-1}=\hat{U}^\dagger$.

This picture is not ideal to derive a QFT in because the language we have developed for field theories explicitly talks about time dependence. Instead, we will use the Heisenberg picture where operators depend on time while states do not. To transform between the two pictures, we will use a unitary operator. The state of our system as it was at $t=0$ obviously does not depend on time which makes it our new state vector

$$|\Psi_H\rangle = \hat{U}^{-1}(t)|\Psi_S(t)\rangle = \hat{U}^\dagger(t)|\Psi_S(t)\rangle = |\Psi_S(0)\rangle\,. \tag{67}$$

To make use of this new picture, we also need to consider how an arbitrary operator $\hat{O}_S$ transforms. It is not difficult to see that

$$\hat{O}_H(t) = \hat{U}^\dagger(t)\hat{O}_S\hat{U}(t) \tag{68}$$

leaves observables invariant. Crucially for what we are about to attempt, this transformation also leaves the commutation relation between $\hat{q}$ and $\hat{p}$ invariant

$$[\hat{q}_H(t),\hat{p}_H(t)] = U^\dagger(t)[\hat{q}_S,\hat{p}_S]U(t) = iU^\dagger(t)U(t) = i\,. \tag{69}$$

This new operator fulfils the EoM if $\hat{O}$ has explicit time dependence through the Heisenberg equation assuming $\hat{O}_H$ has no explicit time dependence

$$i\frac{\mathrm{d}\hat{O}_H}{\mathrm{d}t} = [\hat{O}_H,\hat{H}_H] \equiv \hat{O}_H\hat{H}_H - \hat{H}_H\hat{O}_H\,. \tag{70}$$

[3] Note that for a brief period we will use a hat to indicate that an object is an operator

This is very similar to the classical equivalent Poisson bracket

$$\frac{\mathrm{d}O}{\mathrm{d}t} = \{O, H\} \,. \tag{71}$$

Exercise: Show that (68) indeed leaves expectation values invariant and that the EoM is follows from the Schrödinger equation.

From now on we will always assume the Heisenberg picture and drop in the subscript H.

2.2 Equal-time canonical commutation relations

One of the first things we have learned about QM was that the position and momentum operators do not commute, i.e. $[\hat{q}, \hat{p}] = i$. Extending this for multiple particles, we can define the operator for the position (or momentum) of the i-th particle as $\hat{q}_i$ ($\hat{p}_i$). In position space, these just take the form

$$\hat{q}_k|\Psi(q_1, \cdots, q_n)\rangle = q_k|\Psi(q_1, \cdots, q_n)\rangle \quad \text{and} \quad \hat{p}_k|\Psi(q_1, \cdots, q_n)\rangle = -i\frac{\partial}{\partial q_k}|\Psi(q_1, \cdots, q_n)\rangle \tag{72}$$

The commutation relation is now extended to

$$[\hat{q}_j, \hat{p}_k] = i\delta_{jk} \,, \tag{73a}$$

$$[\hat{q}_j, \hat{q}_k] = [\hat{p}_j, \hat{p}_k] = 0 \,, \tag{73b}$$

since operators acting on different degrees of freedom will always commute. These are referred to as canonical commutation relations (CCRs).

In the continuum limit, we have locations in space $\vec{x}$ instead indices j, field operators $\hat{\phi}(\vec{x}, t)$ instead of coordinate operators $\hat{q}_i$, and conjugate momenta operator $\hat{\pi}(\vec{x}, t)$ instead of momentum operators $\hat{p}_i$. We can now write the CCRs at equal time t

$$[\hat{\phi}(\vec{x}, t), \hat{\pi}(\vec{x}\,', t)] = i\delta^{(3)}(\vec{x} - \vec{x}\,') \,, \tag{74}$$

$$[\hat{\phi}(\vec{x}, t), \hat{\phi}(\vec{x}\,', t)] = [\hat{\pi}(\vec{x}, t), \hat{\pi}(\vec{x}\,', t)] = 0 \,. \tag{75}$$

Note that these are in the Heisenberg picture as they depend on time! While this looks like it might not be Lorentz invariant (after all, we are treating time and space very different here), the formalism we are developing is invariant which will become clearer later.

We will get a better idea of what the operator $\hat{\phi}$ actually does once we write down a solution that satisfies both $\mathcal{L}$ and the CCRs. However, it is helpful to get an idea early on so it is a bit less abstract. Consider the vacuum state $|0\rangle$, i.e. a completely empty universe devoid of any particles (we will formalise this later as well but this suffices for now), as our initial state. We will see shortly that applying $\hat{\phi}(\vec{x}, t)$ on the vacuum will create a new particle at position $\vec{x}$.

We can verify the EoM $i\partial_t\hat{\phi} = [\hat{\phi}, \hat{H}]$ and $i\partial_t\hat{\pi} = [\hat{\pi}, \hat{H}]$ follow from the CCRs. For this, we use that the Hamiltonian is (55)

$$\hat{H} = \int \mathrm{d}^3y \; \hat{\mathcal{H}} = \int \mathrm{d}^3y \Big[\frac{1}{2}\hat{\pi}(\vec{y}, t)^2 + \frac{1}{2}(\vec{\nabla}_y\hat{\phi}(\vec{y}, t))^2 + \frac{1}{2}m^2\hat{\phi}(\vec{y}, t)^2\Big] \tag{76}$$

where we have used that for the KG field $\pi = \dot{\phi}$ which translates to operators $\hat{\pi} = \dot{\hat{\phi}}$. We now calculate commutators

$$[\hat{\phi}(\vec{x}, t), \hat{\mathcal{H}}] = \frac{1}{2}\big[\hat{\phi}(\vec{x}, t), \hat{\pi}(\vec{y}, t)^2\big] + \frac{1}{2}\big[\hat{\phi}(\vec{x}, t), (\vec{\nabla}_y\hat{\phi}(\vec{y}, t))^2\big] + \frac{1}{2}m^2\big[\hat{\phi}(\vec{x}, t), \hat{\phi}(\vec{y}, t)^2\big] \,. \tag{77}$$

Here the derivative in the $\vec{\nabla}_y$ operator is w.r.t. $\vec{y}$ and therefore it commutes with $\hat{\phi}(\vec{x}, t)$. Therefore,

$$[\hat{\phi}(\vec{x}, t), \hat{H}] = \int \mathrm{d}y \; i\hat{\pi}(\vec{y})\delta(\vec{x} - \vec{y}) = i\hat{\pi}(\vec{x}, t) \,, \tag{78}$$

which is fulfilling the EoM as $\hat{\pi} = \dot{\hat{\phi}}$.

Exercise: Show that

$$[\hat{\pi}(\vec{x},t), \hat{H}] = i\Big(\nabla^2\hat{\phi}(\vec{x},t) - m^2\hat{\phi}(\vec{x},t)\Big)\,. \tag{79}$$

Together, we can deduce the KG equation as

$$[[\hat{\phi},\hat{H}],\hat{H}] = i[\hat{\pi},H] = -\Big(\nabla^2\hat{\phi}(\vec{x},t) - m^2\hat{\phi}(\vec{x},t)\Big) = i^2\dot{\hat{\pi}} = -\ddot{\hat{\phi}}\,. \tag{80}$$

2.3 Solution for the field operators

In (63) we have seen a solution $\phi(\vec{x},t)$ to the KG wave equation. To turn this into a solution for the field operator $\hat{\phi}$, we promote the coefficients a and a^* to operators, i.e. $a_{\vec{k}} \to \hat{a}(\vec{k})$ and $a^*_{\vec{k}} \to \hat{a}^\dagger(\vec{k})$. Explicitly, we have for $\hat{\phi}$ and $\hat{\pi}$ (now written in terms of the four-vector x rather than $\vec{x}$ and t)

$$\hat{\phi}(x) = \int \frac{\mathrm{d}^3k}{(2\pi)^3}\sqrt{\frac{1}{2E_{\vec{k}}}}\Big[\hat{a}(\vec{k})\mathrm{e}^{-ik\cdot x} + \hat{a}^\dagger(\vec{k})\mathrm{e}^{ik\cdot x}\Big]\,, \tag{81a}$$

$$\hat{\pi}(x) = \int \frac{\mathrm{d}^3k}{(2\pi)^3}(-i)\sqrt{\frac{E_{\vec{k}}}{2}}\Big[\hat{a}(\vec{k})\mathrm{e}^{-ik\cdot x} - \hat{a}^\dagger(\vec{k})\mathrm{e}^{ik\cdot x}\Big]\,, \tag{81b}$$

where we still have $k^0 = E_{\vec{k}}$.

It is easy to be overwhelmed by the number of symbols in the equation above so let us focus on the relevant terms to schematically write

$$\hat{\phi}(x) \sim \sqrt{\frac{1}{2E_{\vec{k}}}}\Big[\hat{a}(\vec{k}) + \hat{a}^\dagger(\vec{k})\Big]\,, \tag{82a}$$

$$\hat{\pi}(x) \sim (-i)\sqrt{\frac{E_{\vec{k}}}{2}}\Big[\hat{a}(\vec{k}) - \hat{a}^\dagger(\vec{k})\Big]\,. \tag{82b}$$

These equations should look very familiar as they are (up to the $\vec{k}$ dependence), the decomposition of $\hat{x}$ and $\hat{p}$ in terms of ladder operators $\hat{a}$ and $\hat{a}^\dagger$ of the quantum harmonic oscillator. We can find the CCRs for the $\hat{a}$ and $\hat{a}^\dagger$ operators as

$$[\hat{a}(\vec{p}), \hat{a}^\dagger(\vec{q})] = (2\pi)^3\delta^{(3)}(\vec{p}-\vec{q})\,, \tag{83a}$$

$$[\hat{a}(\vec{p}), \hat{a}(\vec{q})] = [\hat{a}^\dagger(\vec{p}), \hat{a}^\dagger(\vec{q})] = 0\,. \tag{83b}$$

Exercise: Show that these follow from the CCRs of $\hat{\phi}$ and $\hat{\pi}$. To do this, inverse-Fourier transform (81) to show that

$$\begin{aligned}\hat{a}(\vec{p}) &= \frac{1}{\sqrt{2E_{\vec{p}}}}\int \mathrm{d}^3x\Big[E_{\vec{p}}\hat{\phi}(x) + i\hat{\pi}(x)\Big]\mathrm{e}^{ip\cdot x}\,,\\ \hat{a}^\dagger(\vec{p}) &= \frac{1}{\sqrt{2E_{\vec{p}}}}\int \mathrm{d}^3x\Big[E_{\vec{p}}\hat{\phi}(x) - i\hat{\pi}(x)\Big]\mathrm{e}^{-ip\cdot x}\,.\end{aligned} \tag{84}$$

Now you can calculate the commutators and reduce them to the CCRs.

The harmonic oscillator became a lot simpler once we wrote the Hamiltonian $\hat{H}$ in terms of the ladder operators. We can do something similar here with one exception

$$
\begin{aligned}
\hat{H} &= \int \mathrm{d}^3x \Big[\frac{1}{2}\hat{\pi}(\vec{x},t)^2 + \frac{1}{2}(\vec{\nabla}\hat{\phi}(\vec{x},t))^2 + \frac{1}{2}m^2\hat{\phi}(\vec{x},t)^2\Big] \qquad (85)\\
&= \int \mathrm{d}^3x \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{\mathrm{d}^3q}{(2\pi)^3}\Bigg(\frac{-\sqrt{E_{\vec{p}}E_{\vec{q}}}}{4}\Big[\hat{a}(\vec{p})\mathrm{e}^{-ip\cdot x} - \hat{a}^\dagger(\vec{p})\mathrm{e}^{ip\cdot x}\Big]\Big[\hat{a}(\vec{q})\mathrm{e}^{-iq\cdot x} - \hat{a}^\dagger(\vec{q})\mathrm{e}^{iq\cdot x}\Big]\\
&\quad - \frac{\vec{p}\cdot\vec{q}}{4\sqrt{E_{\vec{p}}E_{\vec{q}}}}\Big[\hat{a}(\vec{p})\mathrm{e}^{-ip\cdot x} - \hat{a}^\dagger(\vec{p})\mathrm{e}^{ip\cdot x}\Big]\Big[\hat{a}(\vec{q})\mathrm{e}^{-iq\cdot x} - \hat{a}^\dagger(\vec{q})\mathrm{e}^{iq\cdot x}\Big] \qquad (86)\\
&\quad + \frac{m^2}{4\sqrt{E_{\vec{p}}E_{\vec{q}}}}\Big[\hat{a}(\vec{p})\mathrm{e}^{-ip\cdot x} + \hat{a}^\dagger(\vec{p})\mathrm{e}^{ip\cdot x}\Big]\Big[\hat{a}(\vec{q})\mathrm{e}^{-iq\cdot x} + \hat{a}^\dagger(\vec{q})\mathrm{e}^{iq\cdot x}\Big]\Bigg).
\end{aligned}
$$

Let us swap $\vec{p} \to -\vec{p}$ and $\vec{q} \to -\vec{q}$ in the terms that have a positive exponential. Note that this only changes the *spatial* components, i.e. $p \cdot x = E_{\vec{p}} \cdot t - \vec{p} \cdot \vec{x} \to E_{\vec{p}} \cdot t + \vec{p} \cdot x = 2E_{\vec{p}}t - p \cdot x$.

$$
\begin{aligned}
\hat{H} = \int \mathrm{d}^3x \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{\mathrm{d}^3q}{(2\pi)^3}\mathrm{e}^{-i(p+q)\cdot x}&\Bigg(\frac{-\sqrt{E_{\vec{p}}E_{\vec{q}}}}{4}\Big[\hat{a}(\vec{p}) - \hat{a}^\dagger(-\vec{p})\mathrm{e}^{2iE_{\vec{p}}t}\Big]\Big[\hat{a}(\vec{q}) - \hat{a}^\dagger(-\vec{q})\mathrm{e}^{2iE_{\vec{q}}t}\Big]\\
&- \frac{\vec{p}\cdot\vec{q}}{4\sqrt{E_{\vec{p}}E_{\vec{q}}}}\Big[\hat{a}(\vec{p}) + \hat{a}^\dagger(-\vec{p})\mathrm{e}^{2iE_{\vec{p}}t}\Big]\Big[\hat{a}(\vec{q}) + \hat{a}^\dagger(-\vec{q})\mathrm{e}^{2iE_{\vec{q}}t}\Big] \qquad (87)\\
&+ \frac{m^2}{4\sqrt{E_{\vec{p}}E_{\vec{q}}}}\Big[\hat{a}(\vec{p}) + \hat{a}^\dagger(-\vec{p})\mathrm{e}^{2iE_{\vec{p}}t}\Big]\Big[\hat{a}(\vec{q}) + \hat{a}^\dagger(-\vec{q})\mathrm{e}^{2iE_{\vec{q}}t}\Big]\Bigg).
\end{aligned}
$$

We can use that

$$\int \mathrm{d}^3x\ \mathrm{e}^{-i(p+q)\cdot x} = (2\pi)^3\mathrm{e}^{-i(E_{\vec{p}}+E_{\vec{q}})t}\delta^{(3)}(\vec{p}+\vec{q}) \qquad (88)$$

to set $\vec{q} = -\vec{p}$ and $E_{\vec{q}} = E_{\vec{p}}$

$$
\begin{aligned}
\hat{H} &= \int \frac{\mathrm{d}^3p}{(2\pi)^3}\mathrm{e}^{-2iE_{\vec{p}}t}\Bigg(\frac{-E_{\vec{p}}}{4}\Big[\hat{a}(+\vec{p}) - \hat{a}^\dagger(-\vec{p})\mathrm{e}^{2iE_{\vec{p}}t}\Big]\Big[\hat{a}(-\vec{p}) - \hat{a}^\dagger(+\vec{p})\mathrm{e}^{2iE_{\vec{p}}t}\Big]\\
&\qquad + \frac{m^2+\vec{p}\cdot\vec{p}}{4E_{\vec{p}}}\Big[\hat{a}(+\vec{p}) + \hat{a}^\dagger(-\vec{p})\mathrm{e}^{2iE_{\vec{p}}t}\Big]\Big[\hat{a}(-\vec{p}) + \hat{a}^\dagger(+\vec{p})\mathrm{e}^{2iE_{\vec{p}}t}\Big]\Bigg) \qquad (89)\\
&= \int \frac{\mathrm{d}^3p}{(2\pi)^3}\mathrm{e}^{-2iE_{\vec{p}}t}\frac{-E_{\vec{p}}}{4}\Bigg(\Big[\hat{a}(+\vec{p}) - \hat{a}^\dagger(-\vec{p})\mathrm{e}^{2iE_{\vec{p}}t}\Big]\Big[\hat{a}(-\vec{p}) - \hat{a}^\dagger(+\vec{p})\mathrm{e}^{2iE_{\vec{p}}t}\Big]\\
&\qquad - \Big[\hat{a}(+\vec{p}) + \hat{a}^\dagger(-\vec{p})\mathrm{e}^{2iE_{\vec{p}}t}\Big]\Big[\hat{a}(-\vec{p}) + \hat{a}^\dagger(+\vec{p})\mathrm{e}^{2iE_{\vec{p}}t}\Big]\Bigg), \qquad (90)
\end{aligned}
$$

where we have used $\vec{p}\cdot\vec{p} + m^2 = E_{\vec{p}}^2$ since this on-shell condition was always assumed. We now need to multiply out the ladder operators to find

$$\hat{H} = \int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{E_{\vec{p}}}{2}\Big(\hat{a}(+\vec{p})\hat{a}^\dagger(+\vec{p}) + \hat{a}^\dagger(-\vec{p})\hat{a}(-\vec{p})\Big). \qquad (91)$$

Swapping the integration variable $\vec{p} \to -\vec{p}$ in the second term and commuting the $\hat{a}$ to the right of the $\hat{a}^\dagger$

$$\hat{H} = \int \frac{\mathrm{d}^3p}{(2\pi)^3}E_{\vec{p}}\Big(\hat{a}^\dagger(\vec{p})\hat{a}(\vec{p}) + \frac{1}{2}\big[\hat{a}(\vec{p}), \hat{a}^\dagger(\vec{p})\big]\Big). \qquad (92)$$

This is again very similar to the harmonic oscillator where we ended up setting $[\hat{a}, \hat{a}^\dagger] = 1$ and $\hat{H} = \omega(\hat{a}^\dagger\hat{a} + 1/2)$. Here we have a problem though as the commutator evaluates to $\delta(0)$ which is not defined.

2.4 Vacuum energy and normal ordering

What does it mean for $\hat{H}$ to be infinite like this? In practice, nothing. Experiments can only ever measure the difference between energies so that this term will always cancel. However, this is rather unsatisfactory as it does not really solve the underlying problem that our result is infinite, nor does it help us work with the expression. It is quite unwieldy to have infinities like this that are not regulated somehow as it is all too easy to accidentally have the infinity appear "for real" if we for example had actually used the commutation relation rather than just writing the commutator. We need a formal and rigorous way of handling these terms from the beginning.

Lattice regularisation
The simplest way to formalise the removal of this to put our system in a box. Rather than having an infinitely large universe which allows for uncountably infinitely many momenta, we have a finite volume of length L (for example with periodic boundary conditions). This means only discrete momenta are allowed

$$p^i = \frac{2\pi}{L} n^i \qquad \text{with} \qquad n^i = 0, \pm 1, \pm 2, ... \tag{93}$$

and we translate

$$\int \frac{\mathrm{d}^3 p}{(2\pi)^3} \to L^{-3} \sum_{\vec{n}} \tag{94}$$

$$\delta^{(3)}(\vec{p} - \vec{q}) \to \left(\frac{L}{2\pi}\right)^3 \delta_{\vec{p},\vec{q}} \,. \tag{95}$$

Now we have $(2\pi)^3 \delta(0) = L^3$ which is perfectly regular as long as L is finite. In the limit $L \to \infty$, $\hat{H}$ still divergences but we can now define exactly how we want to subtract the $\delta(0)$ problem. Unfortunately, we are still not done because

$$\hat{H} = L^{-3} \sum_{\vec{n}} E_{\vec{p}} \Big(\hat{a}^\dagger(\vec{p}) \hat{a}(\vec{p}) + \frac{1}{2} \underbrace{\big[\hat{a}(\vec{p}), \hat{a}^\dagger(\vec{p})\big]}_{(2\pi)^3 \delta(0)} \Big) = L^{-3} \sum_{\vec{n}} E_{\vec{p}} \, \hat{a}^\dagger(\vec{p}) \hat{a}(\vec{p}) + \sum_{\vec{n}} \frac{E_{\vec{p}}}{2} \,. \tag{96}$$

Even though the L^3 has cancelled, we still have a sum over all possible momenta that will diverge since $E_{\vec{p}} \sim |\vec{p}| \sim \vec{n}$. There are multiple ways of regularising this sum such as using Ramanujan summation (which is common in string theory and would assign $\sum_n n = -1/12$) or lattice regularisation (which defines a largest possible momentum or equivalently a smallest lattice spacing). Whatever way we end up choosing, we can modify our original $\mathcal{L}$ to avoid this problem as a function of the regulator.

This is our first encounter with regularisation (which makes the problem explicit) and renormalisation (which removes the problem at the level of $\mathcal{L}$).

Another more elegant way to deal with this problem is to backtrace a bit. Fundamentally, our problem arises from the commutator of $\hat{a}$ and $\hat{a}^\dagger$ which in turn arises from the $\hat{\phi}$ field operator. In our classical field theory, a and a^* were numbers so their order did not matter. Similarly, when we derived the Hamiltonian $\mathcal{H}$ we wrote ϕ^2 because ϕ is a number. During quantisation, we translated $\phi^2 \to (\hat{\phi})^2$ which lead to the commutator when we decided to move $\hat{a}$ to the right of $\hat{a}^\dagger$. The trick to avoid this is to *always* ensure that $\hat{a}$ is to the right of $\hat{a}^\dagger$ by using normal ordering. In a normal-ordered expression, which we indicate with colons, the operators are defined such that $\hat{a}^\dagger$ always comes before $\hat{a}$ when we go from a classical to a quantum theory. For example

$$: \hat{\phi}(x)^2 : = \int \frac{\mathrm{d}^3 k}{(2\pi)^3 \sqrt{2E_{\vec{k}}}} \Big[\big(\hat{a}(\vec{k})\big)^2 + \big(\hat{a}^\dagger(\vec{k})\big)^2 + 2\hat{a}^\dagger(\vec{k}) \hat{a}(\vec{k}) \Big] \mathrm{e}^{-ik\cdot x} \,, \tag{97}$$

instead of the unordered result

$$\hat{\phi}(x)^2 = \int \frac{\mathrm{d}^3 k}{(2\pi)^3 \sqrt{2E_{\vec{k}}}} \Big[\big(\hat{a}(\vec{k})\big)^2 + \big(\hat{a}^\dagger(\vec{k})\big)^2 + \hat{a}^\dagger(\vec{k}) \hat{a}(\vec{k}) + \hat{a}(\vec{k}) \hat{a}^\dagger(\vec{k}) \Big] \mathrm{e}^{-ik\cdot x} \,. \tag{98}$$

Both of these have the same classical limit but are different operators. However, only the normal-ordered one $:\hat{\phi}^2:$ avoids the problem of the $\delta(0)$ if we define $\hat{H}$ as

$$\hat{H} \to :\hat{H}: = \int \mathrm{d}^3x : \Big[\frac{1}{2}\hat{\pi}(\vec{x},t)^2 + \frac{1}{2}(\vec{\nabla}\hat{\phi}(\vec{x},t))^2 + \frac{1}{2}m^2\hat{\phi}(\vec{x},t)^2\Big] : = \int \frac{\mathrm{d}^3p}{(2\pi)^3} E_{\vec{p}}\ \hat{a}^\dagger(\vec{p})\hat{a}(\vec{p})\,. \tag{99}$$

We can now finally answer the question raised earlier about the interpretation of $\hat{\phi}$ by using the ladder operators $\hat{a}$ and $\hat{a}^\dagger$. Similar to the discussion of the harmonic oscillator, we begin by defining a vacuum state $|0\rangle$ such that $\hat{a}$ destroys it

$$\hat{a}(\vec{p})|0\rangle = 0 \qquad \text{for all possible } \vec{p}\,. \tag{100}$$

and it is properly normalised, i.e.

$$\langle 0|0\rangle = 1\,. \tag{101}$$

The (normal-ordered) energy of this state is just zero since it is immediately destroyed by $\hat{a}$. This is the reason that $|0\rangle$ is called the vacuum state and why the problematic contribution of the $\delta(0)$ to $\hat{H}$ we encountered earlier is called vacuum energy.

2.5 Particles

Now that we know what the vacuum is, what happens if we let $\hat{a}^\dagger$ act upon it? What is the interpretation of the new state

$$|\vec{k}\rangle = C_k \hat{a}^\dagger(\vec{k})|0\rangle\,? \tag{102}$$

This state is clearly orthogonal

$$\langle\vec{p}|\vec{q}\rangle = C_p^* C_q\ \langle 0|\hat{a}(\vec{p})\hat{a}^\dagger(\vec{q})|0\rangle = C_p^* C_q\ \langle 0|[\hat{a}(\vec{p}),\hat{a}^\dagger(\vec{q})]|0\rangle = C_p^* C_q\ (2\pi)^3\delta^{(3)}(\vec{p}-\vec{q})\,. \tag{103}$$

A good and Lorentz-invariant choice of the normalisation is $C_p = \sqrt{2E_{\vec{p}}}$ because then

$$\langle\vec{p}|\vec{q}\rangle = 2E_{\vec{p}}(2\pi)^3\delta^{(3)}(\vec{p}-\vec{q})\,, \tag{104}$$

which is Lorentz invariant.

Lorentz invariance and integrals
Now is a good time to look at this term a bit closer. We will very often see integrals of the form

$$\int \frac{\mathrm{d}^3p}{(2\pi)^3\,2E_{\vec{p}}} f(p)\,, \tag{105}$$

with $E_{\vec{p}} = \sqrt{\vec{p}^2 + m^2}$. Even though it does not appear to be Lorentz-invariant, it is. This means that if $f(p)$ is Lorentz-invariant, so is the integral. Because it is so very common, the measure it is often abbreviated as $\mathrm{d}\Phi_p$, i.e.

$$\int \frac{\mathrm{d}^3p}{(2\pi)^3\,2E_{\vec{p}}} = \int \mathrm{d}\Phi_p \tag{106}$$

To show this, let us begin by adding an additional integration over $p^0 = E_{\vec{p}}$ which is forced on-shell by a delta function

$$\int \mathrm{d}\Phi = \int \frac{\mathrm{d}^4p}{(2\pi)^4}\frac{(2\pi)\delta(p^0 - E_{\vec{p}})}{2E_{\vec{p}}}\,. \tag{107}$$

We can use the identity

$$\delta(\alpha x) = \frac{\delta(x)}{|\alpha|}\,, \tag{108}$$

with $\alpha = 2E_{\vec{p}}$ to write

$$\frac{(2\pi)\delta(p^0 - E_{\vec{p}})}{2E_{\vec{p}}} = (2\pi)\delta\Big(\big(2E_{\vec{p}}\big)\big(p^0 - E_{\vec{p}}\big)\Big) = \theta(p^0)\ (2\pi)\delta\Big(\big(p^0 + E_{\vec{p}}\big)\big(p^0 - E_{\vec{p}}\big)\Big)\,. \tag{109}$$

Here we have used that, as long as $p^0 > 0$ (as enforced by the Heaviside function), $p^0 + E_{\vec{p}} = 2E_{\vec{p}}$. Expanding the argument and using the definition of $E_{\vec{p}}$

$$\int \mathrm{d}\Phi = \int \frac{\mathrm{d}^4 p}{(2\pi)^4}\theta(p^0)\ (2\pi)\delta\big(p^2 - m^2\big)\,. \tag{110}$$

For proper, orthochronous Lorentz transformations, i.e. those that do not change the sign of p^0, this is manifestly Lorentz-invariant.

We can understand what this state is by calculating its energy

$$\begin{aligned}\hat{H}\ |\vec{k}\rangle &= \sqrt{2E_{\vec{k}}}\ \hat{H}\ \hat{a}^\dagger(\vec{k})|0\rangle = \sqrt{2E_{\vec{k}}}\int \frac{\mathrm{d}^3 p}{(2\pi)^3} E_{\vec{p}}\ \hat{a}^\dagger(\vec{p})\hat{a}(\vec{p})\hat{a}^\dagger(\vec{k})|0\rangle \\ &= \sqrt{2E_{\vec{k}}}\int \frac{\mathrm{d}^3 p}{(2\pi)^3} E_{\vec{p}}\ \hat{a}^\dagger(\vec{p})\Big(\big[\hat{a}(\vec{p}), \hat{a}^\dagger(\vec{k})\big] + \hat{a}^\dagger(\vec{k})\hat{a}(\vec{p})\Big)|0\rangle \\ &= \sqrt{2E_{\vec{k}}}\int \frac{\mathrm{d}^3 p}{(2\pi)^3} E_{\vec{p}}\ (2\pi)^3\delta^{(3)}(\vec{p} - \vec{k})\hat{a}^\dagger(\vec{p})|0\rangle = \sqrt{2E_{\vec{k}}}E_{\vec{k}}\hat{a}^\dagger(\vec{k})|0\rangle \\ &= E_{\vec{k}}|\vec{k}\rangle\,, \end{aligned} \tag{111}$$

and momentum

$$\hat{P}^\mu\ |\vec{k}\rangle = k^\mu|\vec{k}\rangle \tag{112}$$

Exercise: Show that the total momentum operator can be written as

$$\hat{P}^\mu = \int \mathrm{d}^3 x\ \ :T^{\mu 0}: = \int \frac{\mathrm{d}^3 p}{(2\pi)^3} p^\mu\ \hat{a}^\dagger(\vec{p})\hat{a}(\vec{p})\,. \tag{113}$$

This means that, as expected, $|\vec{k}\rangle$ is again an eigenstate of $\hat{H}$ but also that $\hat{a}^\dagger(\vec{k})$ has create an excited state with energy $E_{\vec{k}}$ and momentum k^μ. Since $E_{\vec{k}}^2 - \vec{k}^2 = m^2$, it is not unreasonable to say that this is a particle of mass m that has been created with momentum k^μ. Note that this particle is a completely de-localised plane wave because it has definite momentum k^μ. If we instead wanted to create a particle at a given position, we would have to use $\hat{\phi}(x)$ itself.

Note that we can create multiple particles as well. If we apply $\hat{a}^\dagger(\vec{p})$ and $\hat{a}^\dagger(\vec{q})$ we get

$$\sqrt{2E_{\vec{p}}}\hat{a}^\dagger(\vec{p})\ \sqrt{2E_{\vec{q}}}\hat{a}^\dagger(\vec{q})\ |0\rangle \equiv |\vec{p}, \vec{q}\rangle\,. \tag{114}$$

Since the two creation operators commute, $|\vec{p}, \vec{q}\rangle = |\vec{q}, \vec{p}\rangle$. This allows us to conclude that the particles described by the KG equation follow Bose-Einstein statistics, i.e. do not pick up a sign under permutation The spin-statistics theorem, for which no simple proof is known, states that particles following the Bose-Einstein statistics (i.e. bosons) have integer spin while those that follow the Fermi-Dirac statistics (i.e. fermions for which $|\vec{p}, \vec{q}\rangle = -|\vec{q}, \vec{p}\rangle$) have half-integer spin.

2.6 Propagators

The first physics question we can ask of our theory is the amplitude of a particle travelling from y to x. This is an important point from a causality perspective as well: a particle created at y should only be able to reach x if the distance between them is time-like, i.e. $(x - y)^2 > 0$.

The most naive way of phrasing this question is to create a particle at position y and consider the overlap with a particle being created at position x, i.e.

$$D(x-y) = \langle 0|\hat{\phi}(x)\hat{\phi}(y)|0\rangle\,. \tag{115}$$

It is easy to see that

$$D(x-y) = \int \frac{\mathrm{d}^3p}{(2\pi)^3 2E_{\vec{p}}} \mathrm{e}^{-ip\cdot(x-y)} \tag{116}$$

which is Lorentz-invariant as we saw earlier.

> **Exercise**: Show this by working through the operator algebra.

However, this is not quite physical. A particle may propagator from y to x even across space-like separations as long as it does not influence anything outside its future light cone. Therefore we should ask whether the creation at y *influences* the destruction at x, i.e. consider the commutator

$$\langle 0|\big[\hat{\phi}(x), \hat{\phi}(y)\big]|0\rangle = D(x-y) - D(y-x)\,. \tag{117}$$

If $(x-y)^2 < 0$, it is possibly to continuously Lorentz transform $x-y$ into $y-x$ meaning that the expectation value vanishes, ensuring causality[4]. This is not possible for $(x-y)^2 > 0$ so that events that are within each other's light cones can influence each other.

(117) is our first example of a vacuum expectation value (vev) which is an object of the form

$$\langle 0|\hat{O}|0\rangle \tag{118}$$

for some (potentially complicated) operator $\hat{O}$. The term is more commonly used for the vev of a single field $v = \langle 0|\hat{\phi}|0\rangle$. The only field in nature with a non-zero is the Higgs field which has $v = 246\,\mathrm{GeV}$ and is the reason for masses of the W and Z bosons.

Let us study (117) further, now assuming a frame where y happens *before* x, i.e. $x^0 > y^0$, so that we can cleanly speak of y being cause and x being effect. Taking the difference of the two terms using (116)

$$\begin{aligned} \langle 0|\big[\hat{\phi}(x), \hat{\phi}(y)\big]|0\rangle &= \int \frac{\mathrm{d}^3p}{(2\pi)^3 2E_{\vec{p}}} \Big[\mathrm{e}^{-ip\cdot(x-y)} - \mathrm{e}^{ip\cdot(x-y)}\Big] \\ &= \int \frac{\mathrm{d}^3p}{(2\pi)^3} \left(\frac{1}{2E_{\vec{p}}} \mathrm{e}^{-ip\cdot(x-y)}\Big|_{p^0=E_{\vec{p}}} + \frac{1}{-2E_{\vec{p}}} \mathrm{e}^{-ip\cdot(x-y)}\Big|_{p^0=-E_{\vec{p}}} \right). \end{aligned} \tag{119}$$

where we have swapped $\vec{p} \to -\vec{p}$ in the second term and modified $p^0 \to -p^0 = -E_{\vec{p}}$ accordingly. We can use an inverted version of the residue theorem

$$\begin{aligned} \int \frac{\mathrm{d}p^0}{2\pi i} \frac{1}{(p^0)^2 - E_{\vec{p}}^2} \mathrm{e}^{-ip\cdot(x-y)} &= \mathrm{res}_{p^0=+E_{\vec{p}}}\Big(\cdots\Big) + \mathrm{res}_{p^0=-E_{\vec{p}}}\Big(\cdots\Big) \\ &= \frac{1}{2E_{\vec{p}}} \mathrm{e}^{-ip\cdot(x-y)}\Big|_{p^0=E_{\vec{p}}} + \frac{1}{-2E_{\vec{p}}} \mathrm{e}^{-ip\cdot(x-y)}\Big|_{p^0=-E_{\vec{p}}}, \end{aligned} \tag{120}$$

where the contour is defined as shown in Figure 1. This contour is valid only for $x^0 > y^0$ as otherwise the exponential $\mathrm{e}^{-ip^0(x^0-y^0)}$ would diverge. We therefore write

$$D_R(x-y) \equiv \theta(x^0-y^0)\langle 0|\big[\hat{\phi}(x), \hat{\phi}(y)\big]|0\rangle = \int \frac{\mathrm{d}^4p}{(2\pi)^4} \frac{1}{p^2-m^2} \mathrm{e}^{-ip\cdot(x-y)}\,, \tag{121}$$

where we have neatened up the denominator using the definition of $E_{\vec{p}}$. We will call this object the delayed Green's function[5]. A Green's function is the 'inverse' of a differential operator. In our case

$$(\partial^2 + m^2) D_R(x-y) = -i\delta^{(4)}(x-y)\,. \tag{122}$$

[4] This is also true for the commutator itself and does not rely on the vacuum states on either side.

[5] You will find this object often called the retarded Green's function. This term is quite problematic for a number of reasons, not least of all because it is a very archaic term that is not very descriptive.

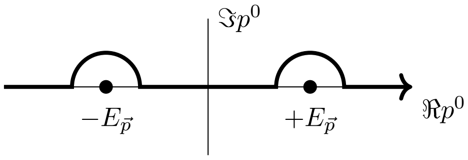


Figure 1: The integration contour used for the delayed propagator D_R

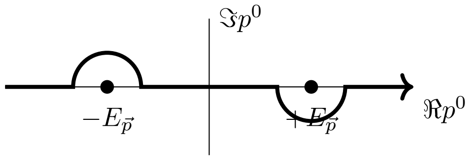


Figure 2: The integration contour used for the Feynman propagator D_F

It turns out that there is a slightly more useful way of defining D which is called the Feynman prescription. Rather than closing the contour above the real axis as in Figure 1, we have the contour weave between the poles as in Figure 2. The result is a mixture of the delayed and advanced Green's functions

$$\begin{aligned} D_F(x-y) = &\begin{cases} D(x-y) & x^0 > y^0 \\ D(y-x) & x^0 < y^0 \end{cases} = \int \frac{\mathrm{d}^4 p}{(2\pi)^4} \frac{1}{p^2 - m^2 + i\epsilon} \mathrm{e}^{-ip\cdot(x-y)} \\ &= \theta(x^0 - y^0)\langle 0|\hat{\phi}(x)\hat{\phi}(y)|0\rangle + \theta(y^0 - x^0)\langle 0|\hat{\phi}(y)\hat{\phi}(x)|0\rangle \equiv \langle 0|T\{\hat{\phi}(x)\hat{\phi}(y)\}|0\rangle \,. \end{aligned} \tag{123}$$

In this Feynman propagator we ensure the order of events using the time-ordering symbol T. We are required to place the last event on the left and the first event on the right. We will see soon why this is a good choice.

3 Interacting scalar field

Now that we understand free quantum fields, we can turn our attention to interacting fields. Unfortunately, there are only a few QFTs that permit an exact analytic solution such as the Schwinger model, a two-dimensional description of a photon and fermion. Most of these are only useful in very limited circumstances or as toy models. Another approach is to solve the theory numerically, usually by placing it on a finite lattice. While this allows the study of realistic models like quantum chromodynamics (QCD), it is extremely complicated to do from scratch.

Here we will instead focus on perturbation theory, i.e. we will consider the interaction to be a small perturbation on top of the free field which we already understand. This means we split the full Lagrangian

$$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_I \tag{124}$$

into a free theory $\mathcal{L}_0$ and an interaction term $\mathcal{L}_I$ which is hopefully small. Perturbation theory allows us to study theories like quantum electrodynamics (QED) or QCD (at least in the high-energy limit). The calculation of the anomalous magnetic moment $g-2$ of the electron I have mentioned at the beginning of the course is (almost completely) done this way. Similarly, calculations that are used at Large Hadron Collider (LHC) and that, for example, helped discover the Higgs boson in 2012, are also mostly done using perturbation theory. This is valid because the coupling strength between different particles, i.e. $\mathcal{L}_I$, is small in these regimes. In the example for $g-2$, we expand in $\alpha_{em} = 1/137$ and the theory value includes effects up to α^5. At the LHC we usually expand in the strong coupling $\alpha_s \approx 0.1$ and a few cutting-edge calculations have reached α_s^3 accuracy.

In this chapter we will only consider the self-interaction of a scalar particle ϕ. Nature has a fundamental scalar particle, namely the Higgs boson, and the theory we develop here can (almost) be used to describe the Higgs boson. The difference is that the real Higgs field is not a single real-valued field (like the one we discussed) but a doublet of two complex-valued fields. The relevant part of the Lagrangian is usually written as (like on that mug)

$$\mathcal{L} = |\partial_\mu \phi|^2 - V(\phi) = |\partial_\mu \phi|^2 + \mu^2 \phi^\dagger \phi - \frac{\lambda}{4!}(\phi^\dagger \phi)^2 \,. \tag{125}$$

We will modify this slightly for our single real-valued scalar field

$$\mathcal{L} = \underbrace{\frac{1}{2}(\partial_\mu \phi)^2 - \frac{1}{2}m^2\phi^2}_{\mathcal{L}_0} - \underbrace{\frac{\lambda}{4!}\phi^4}_{\mathcal{L}_i} \,. \tag{126}$$

I want to stress that you could repeat everything we are about to do for the more general case of the Higgs boson and I would encourage you to try. This theory is often called the ϕ^4 model due to its interaction term. Another related model would be the ϕ^3 theory which would have $\mathcal{L}_i = \lambda/3!\ \phi^3$ which can serve as a simplified model of QED.

It is still possible to derive CCRs for an interacting Lagrangian but we will not be able to write ϕ in terms of ladder operators because this relied on the free field's EoM. The important concept behind perturbative QFT is that the interaction is not just small but usually also short-ranged. This means that at the beginning and end of the experiment, we can consider the particles practically free. For example at the 27 km big LHC which has 25 m big detectors, the actual perturbative interaction takes place in a region that is smaller than 10^{-18} m (based on a typical hard interaction scale of 500 GeV). For $t \to \pm\infty$ the particles involved in the collision are just too far away from each other to feel each other's influence, meaning that we can use the free-particle solution in this regime. We now need to try and find a way to formalise this.

3.1 Interaction picture

In Section 2.1 we have seen the Heisenberg and Schrödinger pictures. In the former, the time dependence fully resides in the operators while in the latter it is in the states. We will now develop a third picture, the interaction picture that splits the time dependence between the two: the operators will follow the free Hamiltonian H_0's Heisenberg equation (70) while the states have a time dependence from the interaction

Hamiltonian $\mathcal{H}_I$. From (124), it follows immediately that

$$H = H_0 + H_I = H_0 - \int \mathrm{d}^3\mathcal{L}_I\,. \tag{127}$$

Recall how we wrote

$$|\Psi_H\rangle = U^{-1}(t)|\Psi_S(t)\rangle = U^\dagger(t)|\Psi_S(t)\rangle = |\Psi_S(0)\rangle\,. \tag{67}$$
$$O_H = U^\dagger(t) O_S U(t)\,, \tag{68}$$

with

$$i\frac{\partial U}{\partial t} = H_S U\,. \tag{66}$$

Rather than taking the full H in (66) we now only use the free H_0 to define

$$|\Psi_I(t)\rangle = \mathrm{e}^{iH_0t}|\Psi_S(t)\rangle \tag{128}$$
$$O_I(t) = \mathrm{e}^{iH_0t}O_S\mathrm{e}^{iH_0t} \tag{129}$$

The time evolution of O_I is still governed by the free Heisenberg equation (70)

$$i\frac{\mathrm{d}O_I}{\mathrm{d}t} = [O_I, H_0]\,, \tag{130}$$

while the states follow a modified Schrödinger equation

$$\begin{aligned} i\frac{\mathrm{d}}{\mathrm{d}t}|\Psi_I(t)\rangle = i\frac{\mathrm{d}}{\mathrm{d}t}\Big(\mathrm{e}^{iH_0t}|\Psi_S(t)\rangle\Big) &= -\mathrm{e}^{iH_0t}H_0|\Psi_S(t)\rangle + \mathrm{e}^{iH_0t}H|\Psi_S(t)\rangle = \mathrm{e}^{iH_0t}H_I\mathrm{e}^{-iH_0t}|\Psi_I(t)\rangle \\ &= \tilde{H}_I|\Psi_I(t)\rangle\,, \end{aligned} \tag{131}$$

where we have used the Schrödinger picture Schrödinger equation for $|\Psi_S(t)\rangle$ and defined the interaction picture interaction Hamiltonian $\tilde{H}_I$.

These rules tell us how to transform from the Schrödinger picture to the interaction picture. We still need to translate between the Heisenberg and interaction pictures. At some instant t_0, we define the pictures to be identical. At any other time t, we translate via the Schrödinger picture

$$\phi_H(t) = U^\dagger(t,t_0)\phi_I(t)U(t,t_0) \qquad \text{with} \qquad U(t,t_0) = \mathrm{e}^{iH_0(t-t_0)}\mathrm{e}^{-iH(t-t_0)}\,. \tag{132}$$

It is easy to see that the time evolution of $U(t,t_0)$ is

$$i\frac{\mathrm{d}}{\mathrm{d}t}U(t,t_0) = \tilde{H}_I U(t,t_0)\,. \tag{133}$$

3.2 The $\mathcal{S}$ matrix

We are now ready to describe a scattering process. The time evolution of the field operator ϕ is given by $U(t,t_0)$. Let us take the limit $t_0 \to -\infty$, well before the scattering takes place. This is where we require the interaction picture and Heisenberg pictures to agree. The field operators in this limit are just the free field which we refer to as the "in" contribution and write in terms of ladder operators

$$\phi_{\text{in}} = \lim_{t\to-\infty}\phi(t) = \int \frac{\mathrm{d}^3k}{(2\pi)^3}\frac{1}{\sqrt{2E_{\vec{k}}}}\Big[a_{\text{in}}(\vec{k})\mathrm{e}^{-ik\cdot x} + a^\dagger_{\text{in}}(\vec{k})\mathrm{e}^{ik\cdot x}\Big]\,. \tag{134}$$

Note that ϕ_{in} still has the time dependence from the free Hamiltonian. This is the case *despite* the limit since there is still the dynamics of the free field. At a later time, including all the way through the scattering, we have

$$\phi(t) = U^\dagger(t,-\infty)\phi_{\text{in}}U(t,-\infty)\,. \tag{135}$$

Experimentally, we do not observe $\phi(t)$ but rather the outcome of the scattering in the far future $t \to \infty$

$$\phi_{\text{out}} = \lim_{t\to+\infty} \phi(t) = U^\dagger(+\infty, -\infty)\phi_{\text{in}} U(+\infty, -\infty)\,. \tag{136}$$

This "out" field is once again free and we can write it again in terms of ladder operators

$$\phi_{\text{out}} = \lim_{t\to+\infty} \phi(t) = \int \frac{\mathrm{d}^3 k}{(2\pi)^3} \frac{1}{\sqrt{2E_{\vec{k}}}} \Big[a_{\text{out}}(\vec{k})\mathrm{e}^{-ik\cdot x} + a^\dagger_{\text{out}}(\vec{k})\mathrm{e}^{ik\cdot x}\Big]\,. \tag{137}$$

Both ϕ_{in} and ϕ_{out} are free field solutions but they are *different* free field solutions. This is because, due to the scattering, the ladder operators a_{in} and a_{out} are different. We refer to these states as the asymptotic states to indicate the limit $t \to \pm\infty$.

The relation between the two sets of asymptotic states is governed by the time evolution operator U. For simplicity, let us define the $\mathcal{S}$ matrix

$$\mathcal{S} = U(+\infty, -\infty) \tag{138}$$

Specifically,

$$a_{\text{out}}(\vec{k}) = \mathcal{S}^\dagger a_{\text{in}}(\vec{k})\mathcal{S} \qquad \text{and} \qquad a_{\text{out}}(\vec{k})^\dagger = \mathcal{S}^\dagger a^\dagger_{\text{in}}(\vec{k})\mathcal{S}\,. \tag{139}$$

This means that $\mathcal{S}$ also transform between in and out states. We begin our experiment with a prepared in state $|\text{in}\rangle_i$ by applying $a^\dagger_{\text{in}}$ on the vacuum. During the scattering this gets transformed into an out state $|\text{out}\rangle_o$ which is made up through $a^\dagger\text{out}$. We will use a subscripts i and o to indicate the ladder operators we have used. These two states are related through the $\mathcal{S}$ matrix

$$|\text{out}\rangle_o = \mathcal{S}^\dagger |\text{in}\rangle_i\,. \tag{140}$$

To understand the scattering process we first need write $|\text{in}\rangle_i$ in terms of the basis of the out states $|n\rangle_o$

$$|\text{in}\rangle_i = \sum_n c_n |n\rangle_o\,. \tag{141}$$

Experimentally we will measure the transition probability between our prepared $|\text{in}\rangle_i$ and a given out state $|n\rangle_o$

$$P \sim \big|\,{}_o\langle n|\text{in}\rangle_i\,\big|^2 = \big|\,{}_i\langle n|\mathcal{S}|\text{in}\rangle_i\,\big|^2 = \big|\,{}_o\langle n|\mathcal{S}|\text{in}\rangle_o\,\big|^2\,. \tag{142}$$

This means our goal will be to find an expression for the $\mathcal{S}$ matrix.

To do this, we would first need to find U to use (138). The definition (132) is not very helpful because of how complicated it is. Instead, we will use the differential equation (133) which defines this solution in the first place. Integrating from $t_0 = -\infty$ to t

$$U(t, -\infty) = 1 - i\int_{-\infty}^{t} \mathrm{d}t_1\ \tilde{H}_I(t_1)\cdot U(t_1, -\infty)\,. \tag{143}$$

Note that, because we fixed $t_0 = -\infty$, the interaction Hamiltonian $\tilde{H}_I$ is defined in terms of in states. Unfortunately, this expression still has a U on the right-hand side so let us iterate this

$$U(t, -\infty) = 1 - i\int_{-\infty}^{t} \mathrm{d}t_1\ \tilde{H}_I(t_1) + (-i)^2 \int_{-\infty}^{t} \mathrm{d}t_1 \int_{-\infty}^{t_1} \mathrm{d}t_2\ \tilde{H}_I(t_1)\cdot\tilde{H}_I(t_2)\cdot U(t_2, -\infty) \tag{144}$$

$$= 1 + \sum_{n=1}^{\infty} (-i)^n \int_{-\infty}^{t} \mathrm{d}t_1 \int_{-\infty}^{t_1} \mathrm{d}t_2 \cdots \int_{-\infty}^{t_{n-1}} \mathrm{d}t_n\ \tilde{H}_I(t_1)\cdot\tilde{H}_I(t_2)\cdots\tilde{H}_I(t_n)\,. \tag{145}$$

Per our construction of the interaction picture, $\tilde{H}_I$ only contains the interaction and not the dynamics of the free field. For example, in the theory we defined in (126), we had $\tilde{H}_I \sim \lambda$. Since we further requested that the coupling λ of interaction is small, we would be justified to assume that $\tilde{H}_I(t_1)\cdot\tilde{H}(t_2) \sim \lambda^2$ is smaller than $\tilde{H}_I(t_1) \sim \lambda$. We therefore often choose to truncate the summation at a finite n (in practice is this rarely more than $n = 2$ or $n = 3$ because of the complexity of the calculation).

One important feature of the iterated integrals in (145) is that we go further into the past in the product of $\tilde{H}_I$ since

$$t_n \leq t_{n-1} \leq \cdots \leq t_1 \leq t\,, \tag{146}$$

which makes for awkward integration boundaries. Instead, let us define the time-ordered product similar to the normal ordering we have used before. Specifically,

$$T\big\{O(t_1)\cdot O(t_2)\cdots O(t_n)\big\} = O(t_{\rho(1)})\cdot O(t_{\rho(2)})\cdots O(t_{\rho(n)})\,, \tag{147}$$

where ρ is the permutation of $\{1, ..., n\}$ such that time is ordered, i.e.

$$t_{\rho(i)} \geq t_{\rho(j)} \qquad \text{if} \qquad i < j\,. \tag{148}$$

We can now change the integration domain to $(-\infty, t]$ for each integral at the cost of a factor of $n!$. Explicitly for $n = 2$, we split the integral into two equal pieces and swap $t_1 \leftrightarrow t_2$

$$\begin{aligned}
&\int_{-\infty}^{t} \mathrm{d}t_1 \int_{-\infty}^{t_1} \mathrm{d}t_2\ \tilde{H}_I(t_1)\cdot\tilde{H}_I(t_2) = \\
&\qquad \frac{1}{2}\int_{-\infty}^{t} \mathrm{d}t_1 \int_{-\infty}^{t_1} \mathrm{d}t_2\ \tilde{H}_I(t_1)\cdot\tilde{H}_I(t_2) + \frac{1}{2}\int_{-\infty}^{t} \mathrm{d}t_2 \int_{-\infty}^{t_2} \mathrm{d}t_1\ \tilde{H}_I(t_2)\cdot\tilde{H}_I(t_1) \\
&\qquad\qquad = \frac{1}{2}\int_{-\infty}^{t} \mathrm{d}t_1 \mathrm{d}t_2\ T\big\{\tilde{H}_I(t_1)\cdot\tilde{H}_I(t_2)\big\}\,.
\end{aligned} \tag{149}$$

Doing the same for all orders, we can rewrite U as

$$U(-\infty, t) = 1 + \sum_{n=1}^{\infty} \frac{(-i)^n}{n!} \int_{-\infty}^{t} \mathrm{d}t_1 \cdots \mathrm{d}t_n\ T\big\{\tilde{H}_I(t_1)\cdot\tilde{H}_I(t_2)\cdots\tilde{H}_I(t_n)\big\} \tag{150}$$

$$\equiv T\exp\left(-i\int_{-\infty}^{t} \mathrm{d}t'\ \tilde{H}_I(t') \right)\,. \tag{151}$$

Here we have introduce the time-ordered exponential as a short-hand. We can make one more simplification by taking the limit $t \to \infty$ and by realising that

$$\tilde{H}_I = \int \mathrm{d}^3x\ \mathcal{H}_I = -\int \mathrm{d}^3x\ \mathcal{L}_I \tag{152}$$

to arrive at our most compact solution for the $\mathcal{S}$ matrix

$$\mathcal{S} = T\exp\left(i\int \mathrm{d}^4x \mathcal{L}_I \right) = T\mathrm{e}^{iS_I} \tag{153}$$

with the interaction part of the action S_I. Looking at this you might think there is some deep interpretation of this expression in terms of the action, similar to the principle of least action we had in the classical case. And you would be right to think this, it is possible to perform the entire quantisation procedure for free and interacting fields by defining the path integral

$$Z = \int \mathcal{D}\phi\ \mathrm{e}^{iS[\phi]/\hbar}\,, \tag{154}$$

which integrates over all possible field configurations ϕ and weights them according to $\mathrm{e}^{iS[\phi]/\hbar}$. In the classical limit $\hbar \to 0$ only the field configuration of the least action contributes to the integral. While very elegant, the path-integral formalism is more complicated since we would have to define what $\mathcal{D}\phi$ means. Therefore, we will not use this method going forward instead relying on the canonical quantisation we have used so far.

3.3 Wick theorem

If we want to make a prediction about a scattering process we need to calculate $\mathcal{S}$ matrix elements to a given order. This means calculating correlators like

$$ {}_i\langle \text{out}| \int \mathrm{d}^4x_1 \cdots \mathrm{d}^4x_n \; T\big\{\mathcal{L}_I(x_1)\cdots\mathcal{L}_I(x_n)\big\}|\text{in}\rangle_i \,. \tag{155} $$

Since both $\mathcal{L}_I$ and the external states involves a number of ϕ fields, we want to be able to calculate general objects like

$$ T\big\{\phi(x_1)\cdots\phi(x_m)\big\}\,. \tag{156} $$

We have already seen a simple case of this with the Feynman propagator $D_F = \langle 0|T\{\phi(x)\phi(y)\}|0\rangle$ in (123). To make calculating this easier, let us define

$$ \phi_I = \int \frac{\mathrm{d}^3k}{(2\pi)^3}\frac{1}{\sqrt{2E_{\vec{k}}}}\Big[\underbrace{a(\vec{k})\mathrm{e}^{-ik\cdot x}}_{\to\phi^{(+)}} + \underbrace{a^\dagger(\vec{k})\mathrm{e}^{+ik\cdot x}}_{\to\phi^{(-)}}\Big] = \phi^{(+)}(x) + \phi^{(-)}(x)\,. \tag{157} $$

This decomposition into positive ($\phi^{(+)}$) and negative ($\phi^{(-)}$) frequency modes is helpful because

$$ \phi^{(+)}|0\rangle = \langle 0|\phi^{(-)} = 0\,. \tag{158} $$

It also makes it easier to define normal ordering which moves the a, and therefore $\phi^{(+)}$, to the right of the $a^\dagger$, and therefore $\phi^{(-)}$. It follows that the vev of a normal-ordered list of fields is zero

$$ \langle 0| \;:\phi(x_1)\cdots\phi(x_m):\; |0\rangle = 0\,. \tag{159} $$

To see why this is so useful, consider again the two-particle case $m=2$ that we considered when defining the Feynman propagator. For $x\neq y$,

$$ \begin{aligned} T\{\phi_I(x)\phi_I(y)\} &= \begin{cases} \phi^{(+)}(x)\phi^{(+)}(y) + \phi^{(+)}(x)\phi^{(-)}(y) + \phi^{(-)}(x)\phi^{(+)}(y) + \phi^{(-)}(x)\phi^{(-)}(y) & x^0>y^0 \\ \phi^{(+)}(y)\phi^{(+)}(x) + \phi^{(+)}(y)\phi^{(-)}(x) + \phi^{(-)}(y)\phi^{(+)}(x) + \phi^{(-)}(y)\phi^{(-)}(x) & x^0<y^0 \end{cases} \\ &= \begin{cases} \phi^{(+)}(x)\phi^{(+)}(y) + \phi^{(-)}(y)\phi^{(+)}(x) + \phi^{(-)}(x)\phi^{(+)}(y) + \phi^{(-)}(x)\phi^{(-)}(y) & x^0>y^0 \\ \phi^{(+)}(y)\phi^{(+)}(x) + \phi^{(-)}(x)\phi^{(+)}(y) + \phi^{(-)}(y)\phi^{(+)}(x) + \phi^{(-)}(y)\phi^{(-)}(x) & x^0<y^0 \end{cases} \\ &\qquad + \begin{cases} [\phi^{(+)}(x),\phi^{(-)}(y)] & x^0>y^0 \\ [\phi^{(+)}(y),\phi^{(-)}(x)] & x^0<y^0 \end{cases}\,. \end{aligned} \tag{160} $$

Every term expect for the commutator is now a normal-ordered product of interaction-picture operators. The commutator is the only term with a non-vanishing vev. Because the interaction-picture fields ϕ follow the time evolution of the *free* Hamiltonian, we can use what we discovered in the previous section. Especially, we can use that the commutator corresponds to the Feynman propagator $D_F(x-y)$.

To simplify our notation, let us define a Wick contraction as

$$ \phi(\overbrace{x)\quad\phi(}y) = \begin{cases} [\phi^{(+)}(x),\phi^{(-)}(y)] & x^0>y^0 \\ [\phi^{(+)}(y),\phi^{(-)}(x)] & x^0<y^0 \end{cases} = D_F(x-y)\,, \tag{161} $$

to indicate which two terms are part of the propagator. Here we drop the I subscript and assume that Wick-contracted terms are always in the interaction picture

$$ T\{\phi(x)\phi(y)\} = \;:\phi(x)\phi(y):\; + \phi(\overbrace{x)\quad\phi(}y)\,. \tag{162} $$

This is the simplest case of the Wick theorem. The more general case is

$$ T\{\phi(x_1)\phi(x_2)\cdots\phi(x_m)\} = \;:\phi(x_1)\phi(x_2)\cdots\phi(x_m) + \text{all possible contractions}\;:\,. \tag{163} $$

To calculate time-ordered products like this we need to sum over all possible ways of grouping fields into pairs using the Wick contractions. As an example, let us consider the case of four fields $m = 4$

$$\begin{aligned} T\{\phi_1\phi_2\phi_3\phi_4\} = :\ & \phi_1\phi_2\phi_3\phi_4 + \wick{\c1\phi_1\c1\phi_2}\phi_3\phi_4 + \wick{\c1\phi_1\phi_2\c1\phi_3}\phi_4 + \wick{\c1\phi_1\phi_2\phi_3\c1\phi_4} \\ & + \phi_1\wick{\c1\phi_2\c1\phi_3}\phi_4 + \phi_1\wick{\c1\phi_2\phi_3\c1\phi_4} + \phi_1\phi_2\wick{\c1\phi_3\c1\phi_4} \\ & + \wick{\c1\phi_1\c1\phi_2\c2\phi_3\c2\phi_4} + \wick{\c1\phi_1\c2\phi_2\c1\phi_3\c2\phi_4} + \wick{\c1\phi_1\c2\phi_2\c2\phi_3\c1\phi_4}\ : , \end{aligned} \tag{164}$$

where we use $\phi_i \equiv \phi(x_i)$ to save space. We already know how to evaluate Wick-contracted terms using D_F, i.e.

$$\begin{aligned} & :\ \phi_1\cdots\phi_{i-1}\cdot\wick{\c1\phi_i\cdot\phi_{i+1}\cdots\phi_{j-1}\cdot\c1\phi_j}\cdot\phi_{j+1}\cdots\phi_m := \\ & D_F(x_i - x_j)\ :\phi_1\cdots\phi_{i-1}\cdot\phi_{i+1}\cdots\phi_{j-1}\cdot\phi_{i+1}\cdots\phi_m : . \end{aligned} \tag{165}$$

If we are considering vevs the uncontracted terms drop out and we only need to consider all m terms contracted, i.e. for example the last line of (164).

Proof of the Wick theorem

We will use a proof by induction. Our base case is $m = 2$ which we have already shown. For the induction step we assume the theorem holds for $m-1$ fields and assume that w.l.o.g. the fields are time-ordered, i.e. $x_1^0 \leq x_2^0 \leq ... \leq x_m^0$. We have

$$T\{\phi_1\cdot\phi_2\cdots\phi_m\} = \phi_1\cdot\phi_2\cdots\phi_m = \left(\phi_1^{(+)} + \phi_1^{(-)}\right)\cdot\ :\phi_2\cdots\phi_m + \Big(\text{contractions} \setminus \phi_1\Big) : , \tag{166}$$

where we have applied the Wick theorem for $\phi_2\cdots\phi_m$. We now need to move the $\phi_1^{(\pm)}$ into the normal ordering. The $\phi_1^{(-)}$ is trivial because it is already where it is supposed to be. To get the $\phi_1^{(+)}$ in we need to commute it all the way through the product. For fields that are already involved in a contraction, this is trivial as D_F is just a number and commutes with everything. This means it is sufficient to only consider uncontracted fields. To simplify the notation a bit, we will write down the case without contractions but the generalisation is trivial once a suitable notation is developed

$$\begin{aligned} \phi_1^{(+)} :\phi_2\cdots\phi_m: \ &= :\phi_2\cdots\phi_m:\ \phi_1^{(+)} + [\phi_1^{(+)},\ :\phi_2\cdots\phi_m:] \\ &= :\phi_1^{(+)}\cdot\phi_2\cdots\phi_m + [\phi_1^{(+)},\phi_2]\cdot\phi_3\cdots\phi_m + \phi_2\cdot[\phi_1^{(+)},\phi_3]\cdot\phi_4\cdots\phi_m + \cdots\ : . \end{aligned} \tag{167}$$

Since $\phi_1^{(+)}$ commutes with the $+$ part of ϕ_i, we have

$$\phi_1^{(+)} :\phi_2\cdots\phi_m:\ = :\phi_1^{(+)}\cdot\phi_2\cdots\phi_m + \wick{\c1\phi_1\c1\phi_2}\cdot\phi_3\cdots\phi_m + \wick{\c1\phi_1\phi_2\cdot\c1\phi_3}\cdot\phi_4\cdots\phi_m + \cdots\ : , \tag{168}$$

which is exactly what we wanted to show.

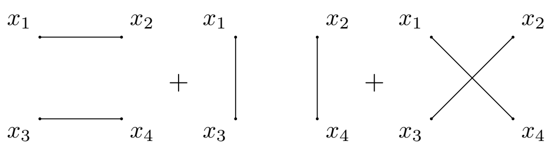
Let us know use what we know to find a graphic representation of these contractions. Consider the vev of (164)

$$\langle 0|T\{\phi_1\phi_2\phi_3\phi_4\}|0\rangle = \langle 0|\Big(\ :\ \wick{\c1\phi_1\c1\phi_2\c2\phi_3\c2\phi_4} + \wick{\c1\phi_1\c2\phi_2\c1\phi_3\c2\phi_4} + \wick{\c1\phi_1\c2\phi_2\c2\phi_3\c1\phi_4}\ :\ +\text{uncontracted}\Big)|0\rangle \tag{169}$$

$$= D_F(x_1 - x_2)D_F(x_3 - x_4) + D_F(x_1 - x_3)D_F(x_2 - x_4) + D_F(x_1 - x_4)D_F(x_2 - x_3) . \tag{170}$$

Remember that the x_i are points in spacetime and $D_F(x_i - x_j)$ contains the part of the amplitude that moves a particle from x_i to x_j (or vice versa). We can therefore draw diagrams to represent these terms

$$\langle 0|T\{\phi_1\phi_2\phi_3\phi_4\}|0\rangle =$$



. (171)

This type of diagram is known as a Feynman diagram and they will soon get more interesting. The three diagrams in (171) merely encode all three ways particles can move between the four positions.

Note that if we had three fields we would have found

$$\langle 0|T\{\phi_1\phi_2\phi_3\}|0\rangle = \langle 0|\Big(:\ \phi_1\phi_2\phi_3 + \overline{\phi_1\phi_2}\phi_3 + \overline{\phi_1\phi_2\phi_3} + \phi_1\overline{\phi_2\phi_3}\ : \Big)|0\rangle = 0\,, \tag{172}$$

because all of the terms have an uncontracted field, i.e. they are $\propto \langle 0|\ :\ \phi_i\ :\ |0\rangle = 0$.

3.4 Asymptotic states and the interacting vacuum

Before we can develop Feynman diagrams further, we need to go to two diversions: one of them relevant, one less so.

The first point is related to the asymptotic states that we defined in (134)

$$\phi_{\text{in}}(x) = \lim_{t\to-\infty}\phi(t) = \int\frac{\mathrm{d}^3k}{(2\pi)^3}\frac{1}{\sqrt{2E_{\vec{k}}}}\Big[a_{\text{in}}(\vec{k})\mathrm{e}^{-ik\cdot x} + a_{\text{in}}^\dagger(\vec{k})\mathrm{e}^{ik\cdot x}\Big]\,. \tag{134}$$

This operator creates a particle at position x. However, it is often more useful to think in momentum space and instead create a particle of momentum p as we did in Section 2.5. For this, we defined (102) for the free field which we translate to the ϕ_{in} fields

$$|p\rangle = \sqrt{2E_{\vec{p}}}\ a_{\text{in}}^\dagger(p)|0\rangle\,. \tag{173}$$

Applying $\phi^{(+)}(x)$ on this state

$$\phi^{(+)}(x)|p\rangle = \int\frac{\mathrm{d}^3k}{(2\pi)^3}\mathrm{e}^{-ik\cdot x}\sqrt{\frac{2E_{\vec{p}}}{2E_{\vec{k}}}}a_{\text{in}}(\vec{k})a_{\text{in}}^\dagger(p)|0\rangle = \int\frac{\mathrm{d}^3k}{(2\pi)^3}\mathrm{e}^{-ik\cdot x}\sqrt{\frac{2E_{\vec{p}}}{2E_{\vec{k}}}}[a_{\text{in}}(\vec{k}), a_{\text{in}}^\dagger(p)]|0\rangle \tag{174}$$

$$= \mathrm{e}^{-ip\cdot x}|0\rangle\,, \tag{175}$$

where we have used that $[a_{\text{in}}, a_{\text{in}}^\dagger]|0\rangle = a_{\text{in}}a_{\text{in}}^\dagger|0\rangle - a_{\text{in}}^\dagger a_{\text{in}}|0\rangle$ and $a_{\text{in}}|0\rangle = 0$. Of course we can repeat the same construction for the out states as well. This is the connection between S matrix elements and the vevs of time-ordered products that we have been calculating with the Wick theorem.

The nature of the vacuum

This diversion is not particularly important for the applications of QFT but is quite a fundamental building block of the theory. So far we have been using $|0\rangle$ both for the ground state of the free theory and of the interacting theory. Unfortunately, since the theories are not the same, there is no reason that the two ground states should be the same (or even that there should be a relation between the two). Here, and only here, we will distinguish between the free theory's vacuum $|0\rangle$ and that of the interacting theory $|\Omega\rangle$.

We define the energy zero as $H_0|0\rangle = 0$ which means that, in general, the energy of the interacting vacuum will be different $E_0 = \langle\Omega|H|\Omega\rangle$. To relate the two Heisenberg states, we apply the full theory's time evolution operator e^{-iHt} to the free vacuum

$$\mathrm{e}^{-iHt}|0\rangle = \sum_n \mathrm{e}^{-iE_nt}|n\rangle\langle n|0\rangle = \mathrm{e}^{-iE_0t}|\Omega\rangle\langle\Omega|0\rangle + \sum_{n>0}\mathrm{e}^{-iE_nt}|n\rangle\langle n|0\rangle \tag{176}$$

with eigenstates $|n\rangle$ and eigenenergies E_n of the full theory which includes the ground state $|\Omega\rangle$. Since per definition, the ground state has the lowest energy $E_n > E_0$ for all $n > 0$. In the limit $t\to\infty(1-i\delta)$ all terms vanish since $\delta > 0$ but the ground state's contribution will vanish the slowest

$$\mathrm{e}^{-iHt}|0\rangle \overset{t\to\infty(1-i\delta)}{=} \mathrm{e}^{-iE_0t}|\Omega\rangle\langle\Omega|0\rangle + \text{terms that vanish faster}\,. \tag{177}$$

We can now solve this for $|\Omega\rangle$ and obtain

$$|\Omega\rangle = \lim_{t\to\infty(1-i\delta)}\Big(\mathrm{e}^{-iE_0t}\langle\Omega|0\rangle\Big)^{-1}\mathrm{e}^{-iHt}|0\rangle\,. \tag{178}$$

To make this a bit easier to use, let us shift $t \to t + t'$ with $t' \ll t$ and add a factor $\mathrm{e}^{iH_0(t+t')}$ which, if applied to $|0\rangle$, will give one

$$|\Omega\rangle = \lim_{t\to\infty(1-i\delta)} \Big(\mathrm{e}^{-iE_0(t+t')}\langle\Omega|0\rangle\Big)^{-1} \mathrm{e}^{-iH(t+t')}|0\rangle \tag{179}$$

$$= \lim_{t\to\infty(1-i\delta)} \Big(\mathrm{e}^{-iE_0(t'-(-t))}\langle\Omega|0\rangle\Big)^{-1} \underbrace{\mathrm{e}^{-iH(t'-(-t))}\mathrm{e}^{iH_0(t+t')}}_{U(t',-t)}|0\rangle\,. \tag{180}$$

This means that we can obtain the interacting vacuum from the free vacuum by evolving it from the distant past ($-t \to \infty$) to the present t'. A similar construction is possible for $\langle\Omega|$ where we need to choose t to be the opposite sign

$$\langle\Omega| = \lim_{t\to\infty(1-i\delta)} \Big(\mathrm{e}^{-iE_0(t-t')}\langle 0|\Omega\rangle\Big)^{-1}\langle 0|U(t,t')\,. \tag{181}$$

We can now write the correlator for $x^0 > y^0 > t'$

$$\begin{aligned}\langle\Omega|\phi(x)\phi(y)|\Omega\rangle = \lim_{t\to\infty(1-i\delta)} & \underbrace{\Big(\mathrm{e}^{-iE_0(t-t')}\langle 0|\Omega\rangle\Big)^{-1}\langle 0|U(t,t')}_{\langle\Omega|} \\ & \underbrace{U^\dagger(x^0,t')\phi_I(x)U(x^0,t')}_{\phi(x)}\underbrace{U^\dagger(y^0,t')\phi_I(y)U(y^0,t')}_{\phi(y)} \\ & \underbrace{U(t',-t)|0\rangle\Big(\mathrm{e}^{-iE_0(t'-(-t))}\langle\Omega|0\rangle\Big)^{-1}}_{|\Omega\rangle}\,.\end{aligned} \tag{182}$$

We can simplify things using $U^\dagger(t_1,t_2) = U(t_2,t_1)$ and $U(t_1,t_2)U(t_2,t_3) = U(t_1,t_3)$ assuming the times are properly ordered

$$\langle\Omega|\phi(x)\phi(y)|\Omega\rangle = \lim_{t\to\infty(1-i\delta)} \Big(\mathrm{e}^{-iE_0\,2t}|\langle\Omega|0\rangle|^2\Big)^{-1}\langle 0|U(t,x^0)\phi_I(x)U(x^0,y^0)\phi_I(y)U(y^0,-t)|0\rangle\,. \tag{183}$$

Assuming $\langle\Omega|\Omega\rangle = 1$, we can write

$$\langle\Omega|\Omega\rangle = \Big(\mathrm{e}^{-iE_0\,2t}|\langle\Omega|0\rangle|^2\Big)^{-1}\langle 0|U(t,t')U(t',-t)|0\rangle \tag{184}$$

to cancel the prefactor phase and E_0

$$\langle\Omega|\phi(x)\phi(y)|\Omega\rangle = \lim_{t\to\infty(1-i\delta)} \frac{\langle 0|U(t,x^0)\phi_I(x)U(x^0,y^0)\phi_I(y)U(y^0,-t)|0\rangle}{\langle 0|U(t,-t)|0\rangle}\,. \tag{185}$$

This is completely time-ordered and would have also held for $y^0 > x^0$. We can therefore write it as a time-ordered product and use (151)

$$\langle\Omega|T\big\{\phi(x)\phi(y)\big\}|\Omega\rangle = \lim_{t\to\infty(1-i\delta)} \frac{\langle 0|T\Big\{\phi_I(x)\phi_I(y)\exp\Big(-i\int_{-t}^{t}\mathrm{d}t'\ \tilde H_I(t')\Big)\Big\}|0\rangle}{\langle 0|T\Big\{\exp\Big(-i\int_{-t}^{t}\mathrm{d}t'\ \tilde H_I(t')\Big)\Big\}|0\rangle}\,. \tag{186}$$

This result is known as the Gell-Mann and Low theorem and it allows us to slightly formalise what we have been doing (which amounts to ignoring the denominator).

Haag's theorem
For the above discussion, we have assumed that both $|0\rangle$ and $|\Omega\rangle$ exist in the same space and that their overlap $\langle 0|\Omega\rangle \neq 0$. In practice, this is not true, making the construction invalid. Further, the operator $U(-\infty, t)$, that we have used to relate the free states to the interacting ones, does not exist either. This result is known as Haag's theorem and it seriously jeopardises the construction of any QFT. Luckily for us, there are a number of ways to, if not rescue the proof, at least stabilise it enough to be used in calculations. This problem is one of the many issues facing a truly axiomatic construction of QFT.

3.5 Feynman diagrams

To study our first non-trivial Feynman let us consider $\langle\Omega|T\{\phi(x)\phi(y)\}|\Omega\rangle$, i.e. the propagator of the interacting theory. We will assume implicitly that these operators are in the interaction picture even though they are not contracted just yet. For this we use the Gell-Mann-Low theorem (186) but will ignore the denominator for now. We have

$$\begin{aligned}\langle\Omega|T\{\phi(x)\phi(y)\}|\Omega\rangle \sim \\ \langle 0|T\left\{\phi(x)\phi(y) - i\phi(x)\phi(y)\int \mathrm{d}t'\ \tilde{H}_I(t') - \frac{1}{2!}\phi(x)\phi(y)\int \mathrm{d}t'\mathrm{d}t''\ \tilde{H}_I(t')H_I(t'') + \cdots\right\}|0\rangle \quad (187)\\ = \langle 0|T\left\{\phi(x)\phi(y) + i\phi(x)\phi(y)\int \mathrm{d}^4z\ \mathcal{L}_I(z) - \frac{1}{2!}\phi(x)\phi(y)\int \mathrm{d}^4z\mathrm{d}^4w\ \mathcal{L}_I(z)\mathcal{L}_I(w) + \cdots\right\}|0\rangle\,. \quad (188)\end{aligned}$$

Here we have substituted in $H_I = \int \mathrm{d}^3z\ \mathcal{L}_I$ and combine the t' integration with the z integration. The first term just corresponds to $D_F(x-y)$. For the second, we write

$$\langle 0|T\left\{i\phi(x)\phi(y)\int \mathrm{d}^4z\ \mathcal{L}_I(z)\right\}|0\rangle = -\frac{i\lambda}{4!}\langle 0|T\left\{\int \mathrm{d}^4z\ \phi(x)\phi(y)\ \phi(z)\phi(z)\phi(z)\phi(z)\right\}|0\rangle\,. \quad (189)$$

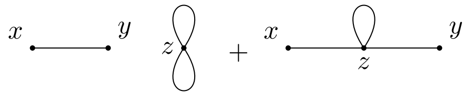
We can now apply Wick's theorem to calculate this vev.

> **Exercise**: Write down all 15 contractions explicitly to show the following

$$\begin{aligned}\langle 0|T\left\{i\phi(x)\phi(y)\int \mathrm{d}^4z\ \mathcal{L}_I(z)\right\}|0\rangle = -\frac{i\lambda}{4!}\Big(3\times D_F(x-y)\int \mathrm{d}^4z\ D_F(z-z)D_F(z-z) \\ + 12\times\int \mathrm{d}^4z\ D_F(x-z)D_F(y-z)D_F(z-z)\Big)\,. \quad (190)\end{aligned}$$

We can visualise this using Feynman diagrams

$$\langle 0|T\left\{i\phi(x)\phi(y)\int \mathrm{d}^4z\ \mathcal{L}_I(z)\right\}|0\rangle = \int \mathrm{d}^4z\Big(\ \cdots\ \Big)\,. \quad (191)$$



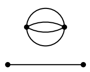
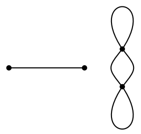
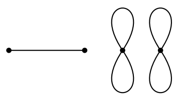
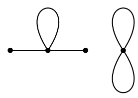
We usually define the symmetry factor, i.e. the 3 and 12 to be included in the diagram. In a Feynman diagram like this, we have propagators (the lines of the diagram) and vertices (points where four lines meet). The vertices are located at positions in spacetime that depend either on the process (x and y) or are integrated over (z). The number of contractions that contribute can grow quite quickly.

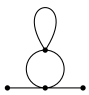
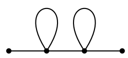
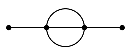
Exercise: Repeat the above for the λ^2 term to show that

$$\begin{aligned}
\frac{1}{2!}\langle 0|T\left\{\phi(x)\phi(y)\int \mathrm{d}^4z\mathrm{d}^4w\ \mathcal{L}_I(z)\mathcal{L}_I(w)\right\}|0\rangle = \frac{\lambda^2}{2!(4!)^2}\int \mathrm{d}^4z\mathrm{d}^4w\Bigg[\Bigg(&4!\ D_F(x-y)\times D_F(z-w)^4 \\
&+(4\cdot 3/2)^2\cdot 2\ D_F(x-y)\times D_F(w-w)D_F(z-z)D_F(w-z)^2 \\
&+3\cdot 3\ D_F(x-y)\times D_F(z-z)^2D_F(w-w)^2\Bigg) \\
&+\Bigg(4\cdot 3\cdot 3\ D_F(x-z)D_F(y-z)\times D_F(w-w)^2D_F(z-z) \\
&+(4\cdot 3)^2\ D_F(x-z)D_F(y-z)\ D_F(z-w)^2D_F(w-w) \\
&+4\cdot 4\cdot 3\cdot 3\ D_F(y-z)D_F(z-z)\ D_F(z-w)\ D_F(w-w)D_F(w-x) \\
&+4\cdot 2\cdot 3\cdot 4\ D_F(y-z)\ D_F(z-w)^3\ D_F(w-x)+(w\leftrightarrow z)\Bigg)\Bigg]\,.
\end{aligned} \tag{192}$$

We can express this using Feynman diagrams as

$$\frac{1}{2!}\langle 0|T\left\{\phi(x)\phi(y)\int \mathrm{d}^4z\mathrm{d}^4w\ \mathcal{L}_I(z)\mathcal{L}_I(w)\right\}|0\rangle = \lambda^2\int \mathrm{d}^4z\mathrm{d}^4w\ \Bigg($$

$\frac{1}{48}$  $+\frac{1}{16}$  $+\frac{1}{128}$  $+\frac{1}{16}$ 

$+\frac{1}{4}$  $+\frac{1}{4}$  $+\frac{1}{6}$  $\Bigg)\,.$ (193)

Here we have written the symmetry factors explicitly even though we previously stated that we consider them part of the diagram. This is for illustration purposes.

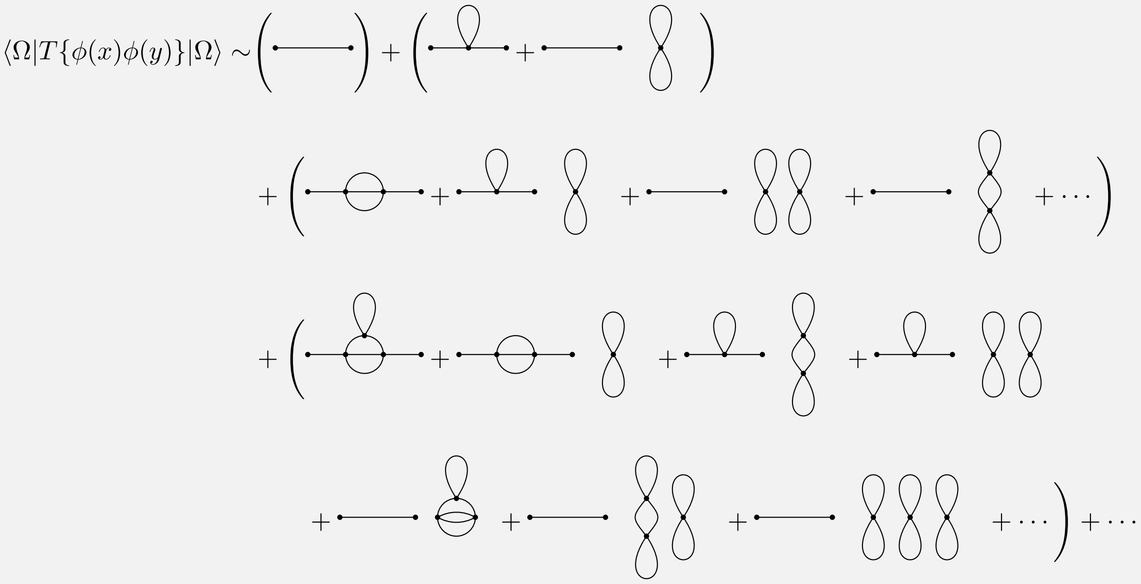
Disconnected pieces and the Gell-Mann-Low denominator

In (191) we encountered two types of Feynman diagrams: a connected diagram where all lines and dots are connected; a disconnected diagram where a diagram was free-floating. Let us focus on the disconnected diagram contribution which evaluated to

z  $$= \int \mathrm{d}^4z\ D_F(z-z)D_F(z-z) \sim \int \mathrm{d}^4z\ (\text{const.}) \sim \text{volume of space time}\,. \tag{194}$$

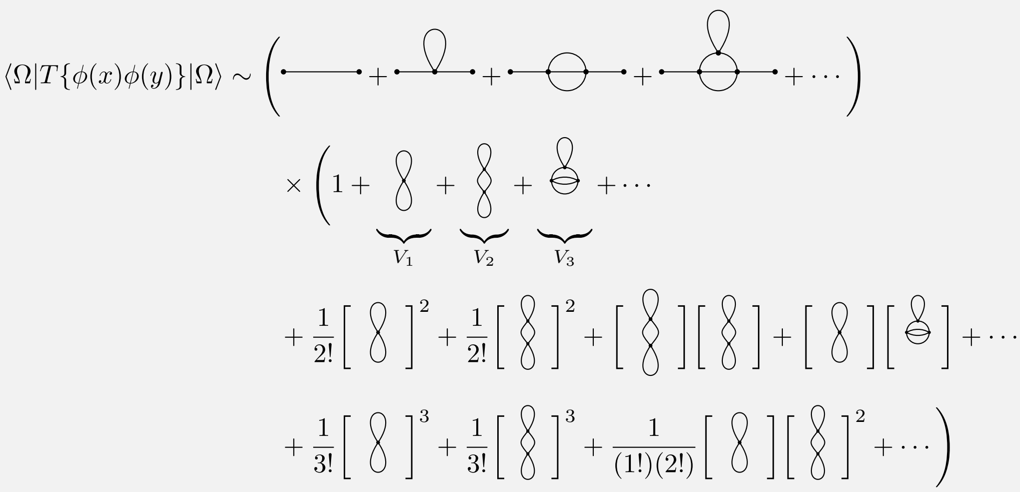
Once again we find ourselves in a situation where our prediction includes the volume of space time which is obviously infinite. For now let us just call this first disconnected piece V_1, the next more complicated disconnected piece is V_2 etc. We will not really worry what these evaluate to because we will soon see that they cancel.

To see what happens, we start with the full series in λ



(195)

Note how in this expression we keep finding the same pieces, both for the connected part and the disconnected ones. We will now try and exploit this structure by rearranging this infinite series by collecting terms not by their power in λ but by their diagrammatic topology. This will lead to us factoring our the connected pieces

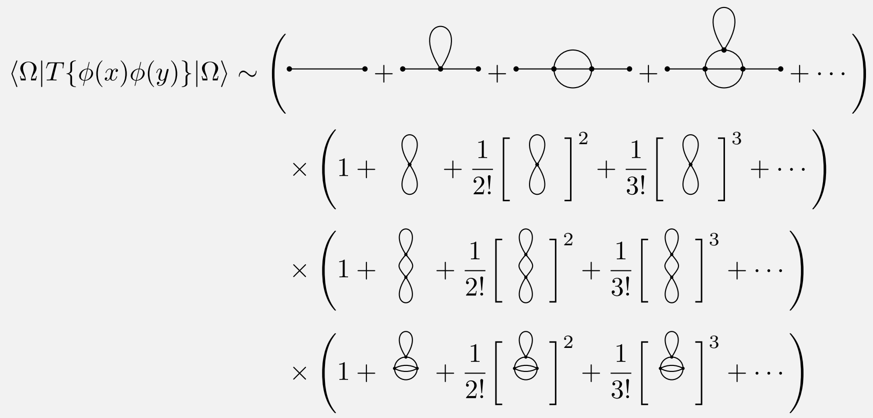


(196)

$$= \left(\sum \text{connected}\right) \times \left(1 + V_1 + V_2 + V_3 + \cdots + \frac{1}{2!}V_1^2 + \frac{1}{2!}V_2^2 + V_1V_2 + V_2V_3 + \cdots \right.$$
$$\left. + \frac{1}{3!}V_1^3 + \frac{1}{3!}V_2^3 + \frac{1}{(1!)(2!)}V_1V_2^2 + \cdots \right) \tag{197}$$

$$= \left(\sum \text{connected}\right) \times \sum_{\text{all }\{n_i\}} \left(\prod_i \frac{1}{n_i!}V_i^{n_i}\right). \tag{198}$$

We can do one more step of rearranging



(199)

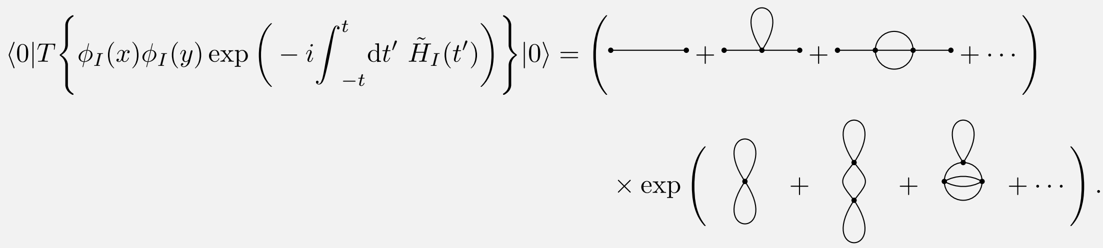
$$= \Big(\sum \text{connected}\Big) \times \Big(1 + V_1 + \frac{1}{2!}V_1^2 + \cdots\Big) \times \Big(1 + V_2 + \frac{1}{2!}V_2^2 + \cdots\Big) \times \cdots \tag{200}$$

$$= \Big(\sum \text{connected}\Big) \times \prod_i \Big(\sum_{n_i} \frac{1}{n_i!} V_i^{n_i}\Big) . \tag{201}$$

This can now be written in terms of an exponential, summing all disconnected diagrams to all orders in λ

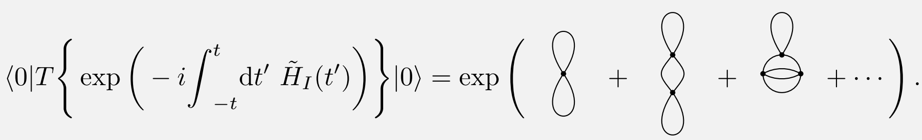
$$\langle\Omega|T\{\phi(x)\phi(y)\}|\Omega\rangle \sim \Big(\sum \text{connected}\Big) \times \prod_i \exp(V_i) = \Big(\sum \text{connected}\Big) \times \exp\Big(\sum_i V_i\Big) . \tag{202}$$

What we have calculated here is the numerator of the Gell-Mann-Low theorem



(203)

We can now consider the denominator which has the same structure but no $\phi(x)$ and $\phi(y)$. We can use the same logic to show that



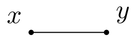
(204)

Therefore, the disconnected contributions cancel and we are left with

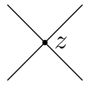
$$\langle\Omega|T\big\{\phi(x)\phi(y)\big\}|\Omega\rangle = \text{sum of all connected diagrams with two external points} . \tag{205}$$

Note that the diagrams we drew in (171) are not disconnected because they are still connected to some external points and therefore do not factor out. This is why the disconnected diagrams are sometimes called vacuum bubbles or vacuum-to-vacuum transitions.

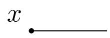
Deriving Feynman diagrams like this is not very efficient. Instead, we usually go the other way around and draw all possible diagrams and then use Feynman rules in position space

For each internal line 

$= D_F(x-y)\,,$ (206a)

For each vertex 

$= -i\lambda \int \mathrm{d}^4 z\,,$ (206b)

For each external line 

$= 1\,,$ (206c)

Divide by the symmetry factor. (206d)

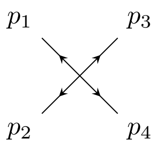
We obtain a factor $1/n!$ from the Taylor expansion which cancels with the $n!$ ways of arranging the n vertices. Further, there are $n!$ ways to arrange the lines going into a vertex which cancels with the $n!$ in $\mathcal{L}_I = \lambda/4!\phi^4$ so that our vertex rule is just λ. After these factors are accounted for we usually still overcounted the diagram. To avoid this, we add the diagram's symmetry factor S which are for example explicitly written in (193) Formally, the symmetry factor is $S = |G|$ the order of the symmetry group G of the diagram that keeps the external lines fixed. A more practical set of rules that will cover almost all use cases is

- lines that start and end in the same vertex, add a factor of 2
- n propagators connecting the same two vertices, add a factor of $n!$
- if two vertices are equivalent, add another factor of 2

The rules as formulated above are valid in position space. Often it is more suitable to have them in momentum space where we write the propagator D_F as a Fourier transform (cf. (123))

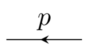
$$D_F(x-y) = \int \frac{\mathrm{d}^4 p}{(2\pi)^4} \frac{1}{p^2 - m^2 + i\epsilon} \mathrm{e}^{-ip\cdot(x-y)}\,. \tag{207}$$

We now assign a momentum p to a propagator and split the factor $\mathrm{e}^{-ip\cdot(x-y)}$ into both ends of the line. This means that for internal vertices we now have

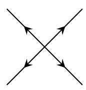


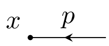
$$\sim \int \mathrm{d}^4 z \mathrm{e}^{ip_1\cdot z}\mathrm{e}^{ip_2\cdot z}\mathrm{e}^{ip_3\cdot z}\mathrm{e}^{ip_4\cdot z} = (2\pi)^4 \delta^{(4)}(-p_1 - p_2 - p_3 - p_4)\,. \tag{208}$$

In other words, momentum is conserved at each vertex. With this, we now have our momentum-space Feynman rules

For each internal line 

$= \frac{i}{p^2 - m^2 + i\epsilon}\,,$ (209a)

For each vertex 
$= -i\lambda\,,$ (209b)

For each external line 

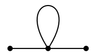
$= \mathrm{e}^{-ip\cdot x}\,,$ (209c)

Impose momentum conservation, (209d)

Integrate over unconstrained momenta $\int \frac{\mathrm{d}^4 p}{(2\pi)^4}\,,$ (209e)

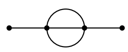
Divide by the symmetry factor. (209f)

Exercise: Calculate the symmetry factors of the following diagrams

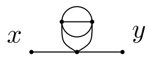


$$S = 2 \tag{210}$$

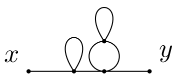
$$S = 16 \tag{211}$$



$$S = 6 \tag{212}$$



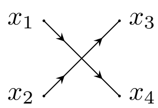
$$S = 12 \tag{213}$$



$$S = 8 \tag{214}$$

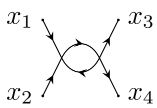
Exercise: Write down all diagrams that contribute to the four-point function $\langle\Omega|T\{\phi_1\phi_2\phi_3\phi_4\}|\Omega\rangle$ up to λ^2 and calculate the amplitude for $\phi(p_1)\phi(p_2) \to \phi(p_3)\phi(p_4)$.

At λ^1 we have a single diagram

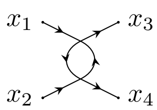


$$= -i\lambda \times \mathrm{e}^{ip_1x_1}\mathrm{e}^{ip_2x_2}\mathrm{e}^{-ip_3x_3}\mathrm{e}^{-ip_4x_4}\delta^{(4)}(p_1+p_2-p_3-p_4) \tag{215}$$

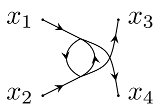
At λ^2, we have four interesting diagrams



$$= (-i\lambda)^2 \int \frac{\mathrm{d}^4k}{(2\pi)^4} \frac{1}{k^2-m^2+i\epsilon} \frac{1}{(k+p_1+p_2)^2-m^2+i\epsilon} \times \text{delta \& exp}\,, \tag{216}$$

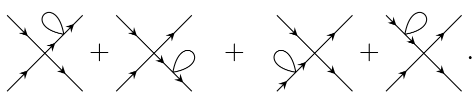


$$= (-i\lambda)^2 \int \frac{\mathrm{d}^4k}{(2\pi)^4} \frac{1}{k^2-m^2+i\epsilon} \frac{1}{(k+p_1-p_3)^2-m^2+i\epsilon} \times \text{delta \& exp}\,, \tag{217}$$



$$= (-i\lambda)^2 \int \frac{\mathrm{d}^4k}{(2\pi)^4} \frac{1}{k^2-m^2+i\epsilon} \frac{1}{(k+p_1-p_4)^2-m^2+i\epsilon} \times \text{delta \& exp}\,. \tag{218}$$

There are also diagrams where a correction is applied to the external line



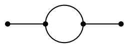
(219)

We will discuss these momentarily.

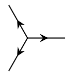
Exercise: Consider the ϕ^3 theory

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\phi)^2 - \frac{1}{2}m^2\mu^2\phi^2 - \frac{\lambda}{3!}\phi^3\,. \tag{220}$$

Calculate the symmetry factor of the following diagram through the Wick theorem

 . (221)

Convince yourself that its Feynman rules are the same as in ϕ^4 except for the vertex

For each vertex  $= -i\lambda\,,$ (222)

Calculate $\langle\Omega|T\{\phi_1\phi_2\phi_3\phi_4\}|\Omega\rangle$ up to λ^1.

3.6 Returning to the $\mathcal{S}$ matrix

To use our formalism to calculate $\mathcal{S}$ matrix elements we would have to repeat the derivation of the Gell-Mann-Low theorem for states other than $|\Omega\rangle$. However, we had to use the fact that the vacuum is the state with the lowest energy which will not be true for any state that contains particles we would like to scatter. It is possible to rescue this argument but how to do this goes well beyond the scope of this course. Instead, the following construction should motivate *why* we might think that we can calculate $\mathcal{S}$ matrix elements using Feynman diagrams.

Consider the $\mathcal{S}$ matrix element (cf. (142)) between an outgoing state f (composed of n particles with momenta $\vec{p}_i$) and an incoming state i (composed of m particles with momenta $\vec{q}_i$)

$$
{}_i\langle f|\mathcal{S}|i\rangle_i = \langle\vec{p}_1\cdots\vec{p}_n|\mathcal{S}|\vec{q}_1\cdots\vec{q}_m\rangle \propto \langle\vec{p}_1\cdots\vec{p}_n|T\Bigg\{\exp\bigg(-i\int_{-t}^{t}\mathrm{d}t'\;\tilde{H}_I(t')\bigg)\Bigg\}|\vec{q}_1\cdots\vec{q}_m\rangle\,. \tag{223}
$$

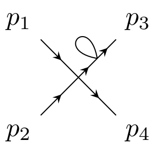
The $\propto$ indicates that we have ignored the equivalent denominator of the Gell-Mann-Low formula (186) in that hopes that it will cancel again if we only consider connected diagrams. Since the external states can be expressed through field operators and the vacuum, we can make them part of the Wick contractions that defined the Feynman diagrams.

However, this will lead to plenty of diagrams where nothing of interest happens like the ones in (171). To avoid this, we define the interesting part of the $\mathcal{S}$ matrix

$$
\mathcal{S} = 1 + i(2\pi)^4\delta^{(4)}(P_i - P_f)\mathcal{T}\,, \tag{224}
$$

where $\mathcal{T}$ contains only the connected diagrams and the 1 all the cases where no interaction takes place. Since our Feynman rules imply momentum conservation, we have made this explicit already here[6].

There is one more restriction on the types of diagrams that enter $\mathcal{T}$. Consider the following diagram which is fully connected



$$
\propto \frac{1}{p'^2 - m^2}\delta^{(4)}(p' - p_3) = \frac{1}{p_3^2 - m^2} = \frac{1}{0}\,, \tag{225}
$$

with the intermediary momentum p'. Momentum conservation forces $p' = p_3$ and because we want p_3 to be on-shell, i.e. $p_3^2 = m^2$, we now have a singularity. This is quite a big problem. $\mathcal{S}$ will only make sense if we exclude this type of diagram where a loop is attached to an external leg like this. One can show that these types of diagrams are similar to the vacuum bubbles we have already excluded in (205). Diagrams that do not have this problem are called amputated.

Let us summarise our achievement

$$
\langle\vec{p}_1\cdots\vec{p}_n|\mathcal{T}|\vec{q}_1\cdots\vec{q}_m\rangle = \sum\Big(\text{amputated \& connected Feynman diagrams}\Big)\,. \tag{226}
$$

There is one final subtly related to this that we will revisit later in Section A.

[6] Sometimes you will see this factor to be defined as part of the $\mathcal{T}$ matrix instead.

4 Cross sections and decay rates

To be able to compare our calculated amplitudes to experimental data, we need to a bit more work. We have (142)

$$P_{i\to f} \sim \left|\langle f|\mathcal{S}|i\rangle\right|^2 \overset{i\neq f}{=} \Big((2\pi)^4\delta^{(4)}(P_i-P_f)\Big)^2 \left|\langle f|\mathcal{T}|i\rangle\right|^2 . \tag{227}$$

The squared delta function is a problem because we only need one of them to solve our integration; the other delta function will then automatically lead to yet another $\delta^{(4)}(0)$. However, we have dealt with this problem before and know to write $(2\pi)^4\delta^{(4)}(0) = V$ with the volume of spacetime V. Therefore, we instead consider the probability *per volume* P/V. We also need to keep in mind that the states require a normalisation $1/\sqrt{E}$. This leads to the probability density

$$\frac{\mathrm{d}P_{i\to f}}{V} = (2\pi)^4\delta^{(4)}(P_i-P_f)\,\left|\langle f|\mathcal{T}|i\rangle\right|^2 \left(\prod_{n=1}^{f}\frac{\mathrm{d}^3p_n}{(2\pi)^3 2E_n}\right)\left(\prod_{n=1}^{i}\frac{1}{2E_n}\right). \tag{228}$$

4.1 Two initial states: cross section

Many particle physics experiments are scattering experiments where we take two particles and collide them. In analogy to classical scattering, we define the cross section of the scattering. Since we are working in a quantum theory rather than a classical one, the cross section describes a probability rather than a physical size.

Consider a cloud of particles of type a at rest with number density ρ_a. Now we shoot a bunch of particles of (a potentially different) type b at the cloud (cf. Figure 3). Along the axis of collision, we have a cross-sectional area A and bunch lengths l_a and l_b. The cross section of the scattering is defined through the number N of scattering events as

$$\sigma = \frac{N}{(\rho_a l_a)(\rho_b l_b)A} = N\underbrace{\frac{A}{N_a\cdot N_b}}_{L^{-1}} . \tag{229}$$

Here we have also defined the total number of a (b) particles N_a (N_b). The combination N_aN_b/A is called the luminosity and it is the reason that the cross section σ is a useful quantity. If we were to repeat our a-b scattering experiment at a different collider which has e.g. more particles in its beams, we would see more events even though the underlying process has not changed. σ encodes the physics, L the parameters of the experiment. This allows us to focus on two-particle scattering and set $\rho_a=\rho_b=1$ even if the real beams may contain as many as 10^{11} particles (the beam intensity of the LHC beams).

For a $2\to f$ process of momenta $p_a, p_b \to p_1, ..., p_f$, we have from (228)

$$\frac{\mathrm{d}P_{a,b\to f}}{V} = \frac{1}{(2E_a)(2E_b)}(2\pi)^4\delta^{(4)}(P_i-P_f)\,\left|\langle f|\mathcal{T}|a,b\rangle\right|^2\left(\prod_{n=1}^{f}\frac{\mathrm{d}^3p_n}{(2\pi)^3 2E_n}\right). \tag{230}$$

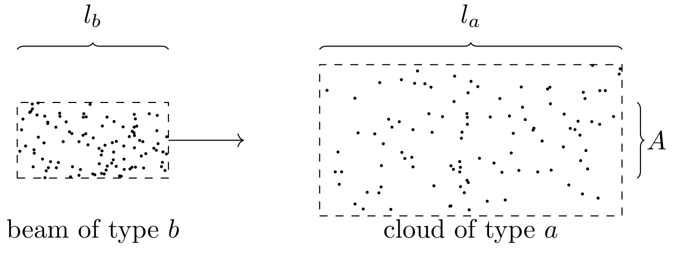


Figure 3: A beam of particles of type b is shot at a cloud of particles of type a. The beam has length l_b and the target l_a. The cross sectional area of the target being hit by the beam is A.

Keep in mind that the volume here is the spacetime volume of the scattering, i.e. $V = t \cdot A \cdot l_a$. For a single scattering, the probability P *is* the number of scattered particles. Therefore the cross section

$$\mathrm{d}\sigma = \frac{\mathrm{d}P_{a,b\to f}}{l_a\, l_b\, A} = \frac{\mathrm{d}P_{a,b\to f}}{V}\frac{t}{l_b}\,. \tag{231}$$

Identifying l_b/t as the velocity of the beam relative to our cloud of particles a, we can now write

$$\mathrm{d}\sigma = \frac{\mathrm{d}P_{a,b\to f}}{V|\vec{v}|} = \frac{1}{(2E_a)(2E_b)|\vec{v}|}\underbrace{\left(\prod_{n=1}^{f}\frac{\mathrm{d}^3p_n}{(2\pi)^3 2E_n}\right)(2\pi)^4\delta^{(4)}\left(p_a+p_b-\sum_{n=1}^{f}p_n\right)}_{\mathrm{d}\Phi_{2\to f}}\underbrace{\big|\langle f|\mathcal{T}|a,b\rangle\big|^2}_{|\mathcal{M}(a,b\to f)|^2}\,. \tag{232}$$

We now need to convince ourselves that $\mathrm{d}\sigma$ is Lorentz invariant since we could otherwise stop a process from happening simply by moving relative to it. We have already seen that the measure $\mathrm{d}^3p/(2E)$ is invariant, making the entire phase space $\mathrm{d}\Phi$ Lorentz invariant. The matrix element $\mathcal{M}(a,b\to f)$ is also fine so that the remaining part is the flux factor $E_aE_b|\vec{v}|$. We an rewrite this in terms of invariants[7]

$$E_aE_b|\vec{v}| = \sqrt{(p_a\cdot p_b)^2 - m_a^2m_b^2} \tag{233}$$

Example numbers from the LHC
When talking about luminosity, we either refer to the instantaneous luminosity L which is measured in $\mathrm{cm}^{-2}\mathrm{s}^{-1}$ or the integrated luminosity $\int L$ which is measured in fb^{-1}. An instantaneous luminosity of $L = 10^{32}\,\mathrm{cm}^{-2}\mathrm{s}^{-1}$ for an entire year corresponds to $\int L = 3.15\,\mathrm{fb}^{-1}$.

In 2024, the integrated luminosity of the LHC was around $\int_{2024} L = 122.6\,\mathrm{fb}^{-1}$. The LHC experiments regularly publish plots that show the luminosity recorded as a function of time over the year. The current version is reproduced in Figure 4. At the time of the Higgs discovery (summer 2012), we had only recorded about $12\,\mathrm{fb}^{-1}$, slightly more than recorded in any given month. You can use the numbers for various cross sections (e.g. $\sigma(pp\to X)\approx 10^{14}\,\mathrm{fb}$ or $\sigma(pp\to H)\approx 5\times 10^4\,\mathrm{fb}$) to find the number of events expected per year. cf. Figure 5.

4.2 One initial state: decay rates

If we have only one particle in the initial state, the only interesting thing that can happen is that this particle decays. We may wonder what the lifetime τ of this particle is or, if it has multiple possible decay channels, what the relative probabilities between the channels is. To do this, we define the decay rate $\Gamma = 1/\tau$ which will be larger for shorter-lived particles (i.e. those with a larger transition probability)

$$\mathrm{d}\Gamma = \frac{\mathrm{d}P_{a\to f}}{V} = \frac{1}{2M}\underbrace{\left(\prod_{n=1}^{f}\frac{\mathrm{d}^3p_n}{(2\pi)^3 2E_n}\right)(2\pi)^4\delta^{(4)}\left(p_a-\sum_{n=1}^{f}p_n\right)}_{\mathrm{d}\Phi_{1\to f}}\underbrace{\big|\langle f|\mathcal{T}|a,b\rangle\big|^2}_{|\mathcal{M}(a\to f)|^2}\,, \tag{234}$$

where we have used that in the rest frame of a the energy $E_a = M$

4.3 Examples

Let us consider a few example of this. We will use a modified version of the ϕ^3 theory we discussed in the previous section that contains two fields ϕ_1 and ϕ_2

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\phi_1)(\partial^\mu\phi_1) - \frac{1}{2}M^2\phi_1^2 + \frac{1}{2}(\partial_\mu\phi_2)(\partial^\mu\phi_2) - \frac{1}{2}m^2\phi_2^2 - \frac{1}{2!}\lambda\phi_1\phi_2^2\,. \tag{235}$$

[7] Note that technically this is only invariant for boosts along the beam axis. For boosts along any other axis, it is not invariant which fits well with our intuition of cross-sectional areas.

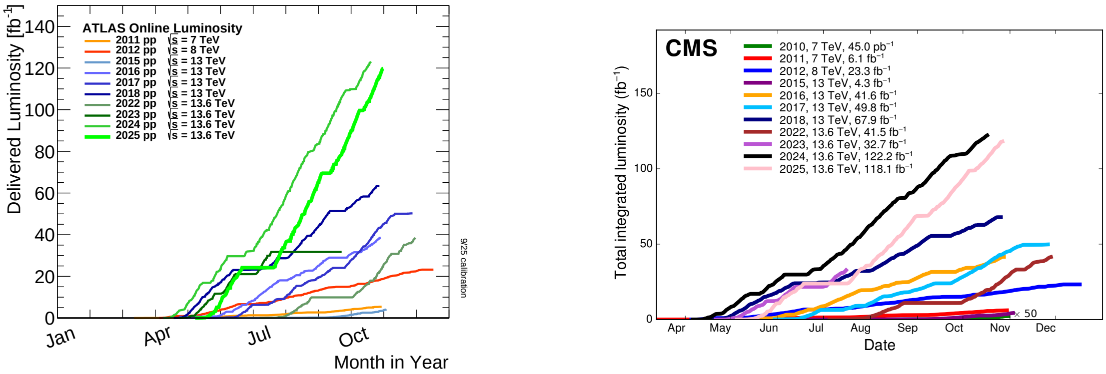


Figure 4: The integrated luminosity recorded by the ATLAS (left) and CMS (right) experiments since 2021. 2024 had a (at the time of writing) record-breaking 122.6 fb^{-1}.

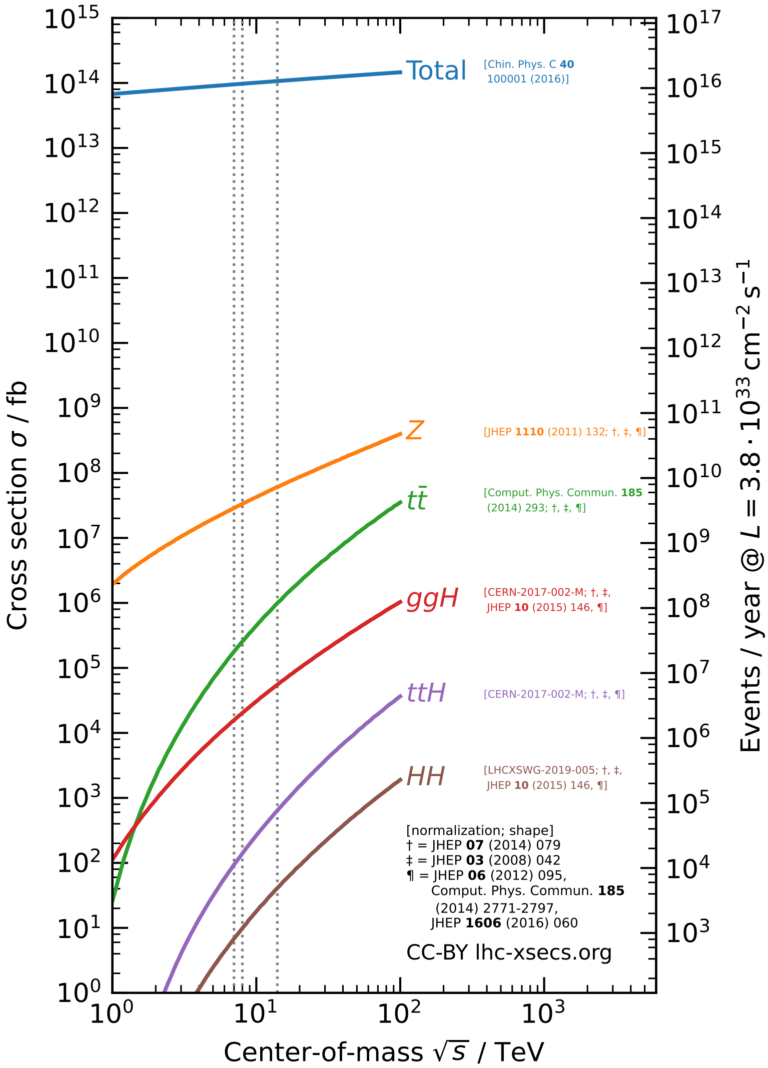
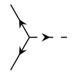
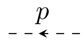
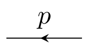


Figure 5: The cross sections of various LHC processes

The Feynman rules of this theory are

For each ϕ_1-ϕ_2-ϕ_2 vertex  $= -i\lambda\,,$ (236)

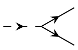
For each internal ϕ_1 line  $= \dfrac{i}{p^2 - M^2 + i\epsilon}\,,$ (237)

For each internal ϕ_2 line  $= \dfrac{i}{p^2 - m^2 + i\epsilon}\,.$ (238)

We could have derived the Feynman rules using the Wick theorem as we did above. Alternatively, we could follow the heuristic of taking a prefactor of $-i$, multiplying the coefficient of the fields ($\lambda/2!$) and multiply with the symmetry factor (1! for the ϕ_1^1 factor and 2! for the ϕ_2^2 factor).

4.3.1 Decay process $\phi_1 \to \phi_2\phi_2$

If we assume that $M > 2m$, the ϕ_1 particle can decay into two ϕ_2 particles. To do this, let us begin by calculating the matrix element $\mathcal{M}$ which is trivial in this case

$\mathcal{M}\big(\phi_1(P) \to \phi_2(p_1)\phi_2(p_2)\big) =$  $= -i\lambda\,.$ (239)

The decay rate therefore is

$$\Gamma(\phi_1 \to \phi_2\phi_2) = \frac{1}{2M}\int (2\pi)^4\delta^{(4)}\Big(P - p_1 - p_2\Big)\frac{\mathrm{d}^3p_1}{(2\pi)^3 2E_1}\frac{\mathrm{d}^3p_2}{(2\pi)^3 2E_2}|\mathcal{M}|^2 \tag{240}$$

$$= \frac{1}{2M}\int \frac{(2\pi)\delta\Big(M - E_1 - E_2\Big)}{2E_1}\frac{\mathrm{d}^3p_2}{(2\pi)^3 2E_2}\lambda^2\,. \tag{241}$$

As always we have $E_i = \sqrt{\vec{p}_i^{\,2} + m^2}$ and since $\vec{p}_1 = -\vec{p}_2$, we have $E_1 = E_2$. We can now write the d^3p_2 integration in spherical coordinates, i.e. $\mathrm{d}^3p_2 = |\vec{p}_2|^2\mathrm{d}|\vec{p}_2|\mathrm{d}\Omega$

$$\Gamma(\phi_1 \to \phi_2\phi_2) = \frac{1}{2M}\int \frac{(2\pi)\delta\big(M - 2E_2\big)}{4E_2^2}\frac{\mathrm{d}\Omega|\vec{p}_2|^2\mathrm{d}|\vec{p}_2|}{(2\pi)^3}\lambda^2\,. \tag{242}$$

Since there is no angular dependence, we can solve the $\mathrm{d}\Omega$ integral and obtain 4π. To solve the $\mathrm{d}|\vec{p}_2|$ integration, we can perform a substitution $\mathrm{d}|\vec{p}_2| = \mathrm{d}E_2 \times E_2/\sqrt{E_2^2 - m^2}$

$$\Gamma(\phi_1 \to \phi_2\phi_2) = \frac{\lambda^2}{8\pi}\int \frac{\mathrm{d}E_2}{E_2}\frac{\sqrt{E_2^2 - m^2}}{M}\delta(M - 2E_2) = \frac{\lambda^2}{16\pi M}\sqrt{1 - 4m^2/M^2}\,. \tag{243}$$

We can identify $\sqrt{1 - 4m^2/M^2}$ as the velocity of one of the ϕ_2 particles since it is defined as

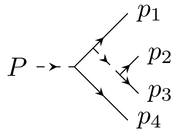
$$\beta = \frac{|\vec{p}_1|}{E_1} = \sqrt{1 - \frac{m^2}{E_1^2}} = \sqrt{1 - \frac{4m^2}{M^2}}\,, \tag{244}$$

with $E_1 = M/2$.

> **Exercise**: Can you explain why we required $M > 2m$ based on this answer? What happens to β for $M < 2m$, for $M = 2m$ or for $M \gg 2m$?

The region $M = 2m$ is referred to as the threshold where the decay is just about possible and the two final-state particles are produced at rest.

Exercise: The decay $\phi_1 \to \phi_2\phi_2\phi_2\phi_2$ is also possible. How would you go about calculating this? The first Feynman diagram would be



$$= (-i\lambda)\frac{i}{(P-p_4)^2 - m^2 + i\epsilon}(-i\lambda)\frac{i}{(p_2+p_3)^2 - M^2 + i\epsilon}\,. \tag{245}$$

There are more diagrams due to the permutations of the outgoing particles. Convince yourself of the above and calculate the full amplitude.

More generally, it is useful to keep

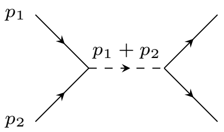
$$\int \mathrm{d}\Phi_2 = \int \mathrm{d}\Omega \frac{1}{16\pi^2}\frac{|\vec{p}_1|}{E_\mathrm{cm}} \tag{246}$$

in mind. Here $E_\mathrm{cm} = M$ and the $\mathrm{d}\Omega$ integral was trivial.

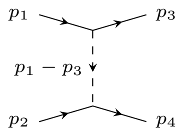
4.3.2 Scattering of $\phi_2\phi_2 \to \phi_2\phi_2$

Let us now calculate the scattering of two ϕ_2 particles. The amplitude $\mathcal{M}$ contains three diagrams

$$\mathcal{M}\big(\phi_2(p_1)\phi_2(p_2) \to \phi_2(p_3)\phi_2(p_4)\big) =$$



$+$



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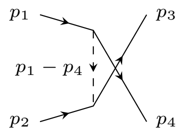


(247)

$$= (-i\lambda)\frac{i}{(p_1+p_2)^2 - M^2}(-i\lambda) + (-i\lambda)\frac{i}{(p_1-p_3)^2 - M^2}(-i\lambda) + (-i\lambda)\frac{i}{(p_1-p_4)^2 - M^2}(-i\lambda)$$

$$= -i\lambda^2\left(\frac{1}{s-M^2} + \frac{1}{t-M^2} + \frac{1}{u-M^2}\right). \tag{248}$$

Here we have defined the Mandelstam variables

$$s = (p_1+p_2)^2 = (p_3+p_4)^2\,,\quad t = (p_1-p_3)^2 = (p_2-p_4)^2\,,\quad u = (p_1-p_4)^2 = (p_2-p_3)^2\,. \tag{249}$$

Exercise: Use momentum conservation to show that $s + t + u = 4m^2$.

The Lorentz-invariant phase space is calculated the same way as before

$$\int \mathrm{d}\Phi_{2\to2} = \int \frac{\mathrm{d}^3p_3}{(2\pi)^3 2E_3}\frac{\mathrm{d}^3p_4}{(2\pi)^3 2E_4}(2\pi)^4\delta^{(4)}\Big(p_1+p_2-p_3-p_4\Big) = \int \mathrm{d}\Omega\frac{1}{16\pi^2}\frac{|\vec{p}_3|}{E_\mathrm{cm}}\,, \tag{250}$$

with $E_\mathrm{cm} = E_1 + E_2$. Unfortunately we now cannot replace $\mathrm{d}\Omega = 4\pi$ because we still have an angular dependency. To see why, let us write down explicit four vectors in the centre-of-mass frame where $\vec{p}_1 + \vec{p}_2 = 0$. For simplicity, we align our coordinate system such that the beam axis is the z direction and that the scattering takes place in the y-z plane

$$p_1 = \big(E, 0, 0, +|\vec{p}_1|\big) = \big(E, 0, 0, +\sqrt{E^2-m^2}\big)\,, \tag{251}$$

$$p_2 = \big(E, 0, 0, -|\vec{p}_2|\big) = \big(E, 0, 0, -\sqrt{E^2-m^2}\big)\,, \tag{252}$$

$$p_3 = \big(E, 0, +\sin\theta|\vec{p}_3|, +\cos\theta|\vec{p}_3|\big) = \big(E, 0, +\sin\theta\sqrt{E^2-m^2}, +\cos\theta\sqrt{E^2-m^2}\big)\,, \tag{253}$$

$$p_4 = \big(E, 0, -\sin\theta|\vec{p}_4|, -\cos\theta|\vec{p}_4|\big) = \big(E, 0, -\sin\theta\sqrt{E^2-m^2}, -\cos\theta\sqrt{E^2-m^2}\big)\,. \tag{254}$$

Since $E \equiv E_1 = \sqrt{|\vec{p}_1|^2 - m^2} = E_2$. For the same reason, $\vec{p}_3 + \vec{p}_4 = 0$ and $E_3 = E_4$. The Mandelstam variables are

$$s = \big(2E, 0, 0, 0\big)^2 = 4E^2 = E_{\text{cm}}^2\,, \tag{255a}$$

$$t = \big(0, 0, -\sin\theta, (+1-\cos\theta)\big)^2 (E^2 - m^2) = -2(E^2-m^2)(1-\cos\theta)\,, \tag{255b}$$

$$u = \big(0, 0, +\sin\theta, (-1-\cos\theta)\big)^2 (E^2 - m^2) = -2(E^2-m^2)(1+\cos\theta)\,. \tag{255c}$$

We can now write the differential cross section as

$$\mathrm{d}\sigma = \frac{1}{(2E)^2|\vec{v}|}\frac{\mathrm{d}\Omega}{16\pi^2}\frac{\sqrt{E^2-m^2}}{2E}\lambda^4\bigg(\frac{1}{s-M^2}+\frac{1}{t-M^2}+\frac{1}{u-M^2}\bigg)^2 \tag{256}$$

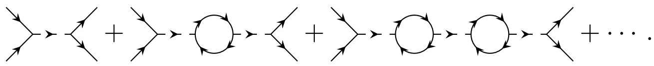
$$= \frac{\mathrm{d}\Omega}{64\pi^2}\frac{\lambda^4}{E^2(M^2-4E^2)^2}\left(\frac{14E^4 - 2m^4 + 4m^2(2M^2-3E^2) - 3M^4 + 2(m^2-E^2)^2\cos(2\theta)}{(2E^2-2m^2+M^2)^2 - 4(m^2-E^2)^2\cos^2\theta}\right)^2. \tag{257}$$

If we set $M = m = 0$, we find a very short expression

$$\mathrm{d}\sigma = \frac{\mathrm{d}\Omega}{1024\pi^2}\frac{\lambda^4}{E^6}\frac{(3+\cos^2\theta)^2}{(1-\cos^2\theta)^2}\,. \tag{258}$$

What do we now do with this object? We can either visualise the differential distribution $\mathrm{d}\sigma/\mathrm{d}\Omega$, normally written as $\mathrm{d}\sigma/\mathrm{d}(\cos\theta)$ or $\mathrm{d}\sigma/\mathrm{d}\theta$. Alternatively, we could integrate over $\mathrm{d}\Omega$ and obtain the full cross section of this process occurring.

A few comments are in-order. From (255) it is clear that $s = 4E^2 \geq 4m^2 > 0$ and $t, u \leq 0$. This means the second and third term in (256) will not be a problem as long $M \neq 0$. However, $s = M^2$ is allowed and the cross section would explode if we picked this value of s. This is fixed by adding higher-order corrections to the ϕ_1 propagator, i.e.



(259)

Even though each term is progressively more suppressed by λ, the addition of more propagators $1/(s-M^2)$ makes up for this and we need to calculate infinitely many such insertions. We will come back to this in Section 7 and Appendix A.

We should also note the behaviour of (258) for $\cos\theta \to \pm 1$. This corresponds to a scattering angle of $\theta = 0$ or $\theta = \pi$, i.e. when the outgoing particles follow the beam axis. Since the two ϕ_2 particles are indistinguishable, this just means that the particles pass each other without interacting. The divergence we see here is to the fact that we have split the $\mathcal{S}$ matrix as $\mathcal{S} = 1 + \mathcal{T}$ in (224). When we constructed $\mathcal{T}$ we assumed that the initial and final state were different which is not the case for $\cos\theta = \pm 1$.

5 Fermions and Photons

The scalar particles we have studied so far had spin 0. What about higher spins? One can show a fundamental particle must be in the fundamental representation of the Lorentz group $SO^+(1,3)$ which limits possible spins to

- $j = 0$: Higgs boson but also e.g. pion, Helium-4, Carbon-12;
- $j = 1/2$: quarks and leptons but also e.g. proton, neutron;
- $j = 1$: vector bosons like photons, gluons, Z and W, but also deuteron, Nitrogen-14;
- $j = 3/2$: called a Rarita-Schwinger particle, no fundamental example has been discovered but composite particles like Lithium-7 or the Δ^{++} exist;
- $j = 2$: any massless $j = 2$ particle can be shown to be a graviton of which we unfortunately do not have a consistent theory.

5.1 Fermions

Since fermions are particles of matter, let us consider them first. The main problem with the KG equation are its negative energy solutions. If viewed not in the context of a QFT but as a non-relativistic quantum theory, this would mean that no ground state could exist since it could always have less energy. These solutions appear because the KG is quadratic in ∂_t. This in turn was a consequence of the on-shell relation $p^2 = m^2$. Dirac's approach was then to write the KG operator as a product

$$-(\partial^2 + m^2) = (i\gamma^\nu\partial_\nu + m)(i\gamma^\mu\partial_\mu - m)\,. \tag{260}$$

If we therefore choose our differential equation to be

$$(i\gamma^\mu\partial_\mu - m)\psi = 0\,, \tag{261}$$

it is by construction linear in ∂_t, manifestly Lorentz invariant and fulfils the on-shell condition. Unfortunately, the γ^μ cannot be a mere number since we require

$$\gamma^\mu\gamma^\nu\partial_\mu\partial_\nu = \eta^{\mu\nu}\partial_\mu\partial_\nu\,. \tag{262}$$

Note that this does not mean that $\gamma^\mu\gamma^\nu = \eta^{\mu\nu}$ since the tensor $\partial_\mu\partial_\nu$ is symmetric under μ-ν exchange. Instead, we can write with the anti-commutator $\{A, B\} = AB + BA$

$$\gamma^\mu\gamma^\nu\partial_\mu\partial_\nu = \frac{1}{2}\Big(\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu\Big)\partial_\mu\partial_\nu = \frac{1}{2}\{\gamma^\mu, \gamma^\nu\}\partial_\mu\partial_\nu\,. \tag{263}$$

This is now the defining property of the γ matrices

$$\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}\,. \tag{264a}$$

To ensure that the Hamiltonian is self-adjoint, we also require the following normalisation

$$\big(\gamma^0\big)^2 = 1 \qquad \text{and} \qquad \big(\gamma^k\big)^2 = -1 \quad \text{with} \quad k = 1, 2, 3\,. \tag{264b}$$

These properties are actually sufficient for anything we may want to use γ matrices for, even without writing them down as explicit 4×4 objects.

Note that γ^μ is a Lorentz vector, i.e. a list of four 4×4 matrices. This can be made a bit clearer when using indices for this spinor space. For example with the identity matrix in spinor space I

$$(261) = \sum_{b=1}^{4}(i\gamma^\mu_{ab} - m\,I_{ab})\psi_b = 0\,, \tag{265}$$

$$(268a) = \sum_{\mu=0}^{3}\sum_{b=1}^{4}\gamma^\mu_{ab}\gamma_{\mu,bc} = 4\,I_{ac}\,, \tag{266}$$

$$(268b) = \sum_{\mu=0}^{3}\sum_{b,c=1}^{4}\gamma^\mu_{ab}\gamma^\nu_{bc}\gamma_{\mu,cd} = -2\gamma^\nu_{ad}\,. \tag{267}$$

We will usually not write spinor indices and reserve the $\cdot$ operator for a Lorentz or three-vector product.

Exercise: Show the following identities

$$\gamma^\mu \gamma_\mu = 4\,, \tag{268a}$$
$$\gamma^\mu \gamma^\nu \gamma_\mu = -2\gamma^\nu\,, \tag{268b}$$
$$\gamma^\mu \gamma^\nu \gamma^\rho \gamma_\mu = 4\eta^{\nu\rho}\,, \tag{268c}$$
$$\gamma^\mu \gamma^\nu \gamma^\rho \gamma^\sigma \gamma_\mu = -2\gamma^\sigma \gamma^\rho \gamma^\nu\,. \tag{268d}$$

For example

$$\gamma^\mu \gamma_\mu = \eta_{\mu\nu} \gamma^\mu \gamma^\nu \overset{*}{=} \frac{1}{2} \eta_{\mu\nu} (\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu) = \eta_{\mu\nu} \eta^{\mu\nu} = 4\,, \tag{269}$$

where we have used at $*$ that the η tensor is symmetric. For the next relation, we write using the anti-commutator and the previous result

$$\gamma^\mu \gamma^\nu \gamma_\mu = (2\eta^{\mu\nu} - \gamma^\nu \gamma^\mu)\gamma_\mu = 2\gamma^\nu - 4\gamma^\nu = -2\gamma^\nu\,. \tag{270}$$

The others follow exactly the same way.

Exercise: Using the fact that $\mathrm{tr}(aA + bB) = a\mathrm{tr}(A) + b\mathrm{tr}(B)$ and that $\mathrm{tr}(A \cdot B \cdots C \cdot D) = \mathrm{tr}(D \cdot A \cdot B \cdots C)$, show that

$$\mathrm{tr}(\gamma^\mu) = 0\,, \tag{271a}$$
$$\mathrm{tr}(\underbrace{\gamma^{\mu_1} \cdots \gamma^{\mu_k}}_{\text{odd}}) = 0\,, \tag{271b}$$
$$\mathrm{tr}(\gamma^\mu \gamma^\nu) = 4\eta^{\mu\nu}\,, \tag{271c}$$
$$\mathrm{tr}\Big(\gamma^\mu \gamma^\nu \gamma^\rho \gamma^\sigma\Big) = 4\Big(\eta^{\mu\nu}\eta^{\rho\sigma} - \eta^{\mu\rho}\eta^{\nu\sigma} + \eta^{\mu\sigma}\eta^{\nu\rho}\Big)\,. \tag{271d}$$

We begin with

$$\mathrm{tr}(\gamma^\mu) = \frac{1}{-2}\mathrm{tr}(\gamma^\nu \gamma^\mu \gamma_\nu) = \frac{1}{-2}\mathrm{tr}(\gamma_\nu \gamma^\nu \gamma^\mu) = \frac{4}{-2}\mathrm{tr}(\gamma^\mu) \tag{272}$$

which can only be satisfied if the trace is zero. Similarly, we can show γ^5 (cf. next section)

$$\begin{aligned}\mathrm{tr}(\gamma^{\mu_1}\gamma^{\mu_2}\cdots\gamma^{\mu_n}) &= \mathrm{tr}(\gamma^{\mu_1}\gamma^{\mu_2}\cdots\gamma^{\mu_n}\gamma^5\gamma^5) = \mathrm{tr}(\gamma^5\gamma^{\mu_1}\gamma^{\mu_2}\cdots\gamma^{\mu_n}\gamma^5) = -\mathrm{tr}(\gamma^{\mu_1}\gamma^5\gamma^{\mu_2}\cdots\gamma^{\mu_n}\gamma^5) \\ &= +\mathrm{tr}(\gamma^{\mu_1}\gamma^{\mu_2}\gamma^5\cdots\gamma^{\mu_n}\gamma^5) = (-1)^n\mathrm{tr}(\gamma^{\mu_1}\gamma^{\mu_2}\cdots\gamma^{\mu_n}\gamma^5\gamma^5)\,.\end{aligned} \tag{273}$$

If n is odd, this means $\mathrm{tr}(\cdots) = -\mathrm{tr}(\cdots)$ which is only satisfied if the trace vanishes. Next,

$$\mathrm{tr}(\gamma^\mu\gamma^\nu) = \frac{1}{2}\Big(\mathrm{tr}(\gamma^\mu\gamma^\nu) + \mathrm{tr}(\gamma^\nu\gamma^\mu)\Big) = \frac{1}{2}\mathrm{tr}\Big(\{\gamma^\mu, \gamma^\nu\}\Big) = \eta^{\mu\nu}\mathrm{tr}(1)\,, \tag{274}$$

with $\mathrm{tr}(1) = 4$.

Exercise: Finally, show that

$$(\gamma^\mu)^\dagger = \gamma^0 \gamma^\mu \gamma^0\,. \tag{275}$$

We can write the Dirac equation (261) using a Hamiltonian

$$i\frac{\partial\psi}{\partial t} = \underbrace{\Big(-i\gamma^0\vec{\gamma}\cdot\vec{\nabla} + \gamma^0 m\Big)}_{H}\,\psi\,. \tag{276}$$

Since we want $H = H^\dagger$, we need $(\gamma^0\vec{\gamma})$ and (γ^0) the be self-adjoint as well, justifying (264b).

5.1.1 Pauli's fundamental theorem and basis of γ matrices

The γ matrices are fully defined through (264), i.e. we may use any set of matrices that fulfil these requirements. This means that if we have a different set of matrices $(\gamma')^\mu$ that also fulfil (264), they must be related to γ^μ through a constant invertible matrix S

$$(\gamma')^\mu = S^{-1}\gamma^\mu S\,. \tag{277}$$

This is called Pauli's fundamental theorem. Its proof is not that important but it makes use of an important fact: we can write any product of γ matrices using a basis of $16 = 4\times 4$ elements.

To pick these, it is customary to define a fifth γ matrix

$$\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3 = -\frac{i}{4!}\varepsilon^{\mu\nu\rho\sigma}\gamma_\mu\gamma_\nu\gamma_\rho\gamma_\sigma\,, \tag{278}$$

with the totally anti-symmetric tensor $\epsilon^{\mu\nu\rho\sigma}$, defined to be $\varepsilon^{0123} = 1$.

Exercise: Show using the anti-commutation relations and the definition of γ^5

$$(\gamma^5)^\dagger = \gamma^5\,, \tag{279a}$$

$$(\gamma^5)^2 = 1\,, \tag{279b}$$

$$\{\gamma^5, \gamma^\mu\} = 0\,. \tag{279c}$$

For example, for the anti-commutator

$$\{\gamma^5, \gamma^\lambda\} = -\frac{i}{4!}\varepsilon_{\mu\nu\rho\sigma}\{\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma, \gamma^\lambda\}\,, \tag{280}$$

we write

$$\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\gamma^\lambda = \gamma^\mu\gamma^\nu\gamma^\rho 2\eta^{\lambda\sigma} - \gamma^\mu\gamma^\nu\gamma^\sigma 2\eta^{\lambda\rho} + \gamma^\mu\gamma^\rho\gamma^\sigma 2\eta^{\nu\lambda} - \gamma^\nu\gamma^\rho\gamma^\sigma 2\eta^{\mu\lambda} + \gamma^\lambda\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\,. \tag{281}$$

Therefore

$$\{\gamma^5, \gamma^\lambda\} = -\frac{2i}{4!}\varepsilon_{\mu\nu\rho\sigma}\Big(\gamma^\mu\gamma^\nu\gamma^\rho\eta^{\lambda\sigma} - \gamma^\mu\gamma^\nu\gamma^\sigma\eta^{\lambda\rho} + \gamma^\mu\gamma^\rho\gamma^\sigma\eta^{\nu\lambda} - \gamma^\nu\gamma^\rho\gamma^\sigma\eta^{\mu\lambda} + \gamma^\lambda\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\Big) \tag{282}$$

We can write any product of γ matrices using the following basis

$$\Gamma = \Big\{\underbrace{1}_{1}, \underbrace{\gamma^\mu}_{4}, \underbrace{\tfrac{i}{2}[\gamma^\mu,\gamma^\nu]}_{6}, \underbrace{\gamma^5}_{1}, \underbrace{\gamma^\mu\gamma^5}_{4}\Big\}\,. \tag{283}$$

The numbers indicate that the number of basis elements of this form. We refer to these as scalar, vector, tensor, pseudo-scalar and pseudo-vector respectively.

If we simultaneously transform the spinor ψ as $\psi \to S^{-1}\psi$, the Dirac equation transforms to

$$(i\gamma\cdot\partial - m)\psi \to (i\gamma'\cdot\partial' - m)\psi' = (i(S^{-1}\gamma S)\cdot\partial - m)S^{-1}\psi = (i\gamma\cdot\partial - m) \tag{284}$$

This means physics will always be invariant under basis change.

Even if the exact form of the γ matrices does not matter, it is sometimes helpful to have one

$$\gamma^0 = \begin{pmatrix} 1_{2\times 2} & 0_{2\times 2} \\ 0_{2\times 2} & -1_{2\times 2} \end{pmatrix}, \qquad \gamma^i = \begin{pmatrix} 0_{2\times 2} & \sigma^i \\ -\sigma^i & 0_{2\times 2} \end{pmatrix}, \tag{285}$$

with the Pauli matrices σ^i

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{286}$$

5.1.2 Transformations of the Dirac equation

If we want to view the spinor ψ as physically meaningful, we need to understand how it transforms in different frames. Consider therefore two frames described by x and x' with $(x')^\mu = \Lambda^\mu{}_\nu x^\nu$. If $\psi(x)$ is a solution in the x frame and $\psi'(x')$ is a solution in the x' frame, we must have

$$\left(i\gamma^\mu \frac{\partial}{\partial x^\mu} - m\right)\psi(x) = \left(i\gamma^\mu \frac{\partial}{\partial (x')^\mu} - m\right)\psi'(x') = 0\,. \tag{287}$$

We can use (27) to transform the derivative

$$0 = \left(i\gamma^\mu \frac{\partial}{\partial x^\mu} - m\right)\psi(x) = \left(i\underbrace{\gamma^\mu \Lambda^\nu{}_\mu}_{(\gamma')^\nu} \frac{\partial}{\partial (x')^\nu} - m\right)\psi(\Lambda^{-1}x')\,. \tag{288}$$

The new γ matrices $(\gamma')^\mu = \Lambda^\mu{}_\nu \gamma^\nu$ still need to fulfil (264a)

$$\{(\gamma')^\alpha, (\gamma')^\beta\} = \{\Lambda^\alpha{}_\mu \gamma^\mu, \Lambda^\beta{}_\nu \gamma^\nu\} = \Lambda^\alpha{}_\mu \Lambda^\beta{}_\nu \{\gamma^\mu, \gamma^\nu\} = \Lambda^\alpha{}_\mu \Lambda^\beta{}_\nu 2\eta^{\mu\nu} = 2\eta^{\alpha\beta}\,, \tag{289}$$

since (23) guarantees $\eta_{\mu\nu}\Lambda^\mu{}_\rho \Lambda^\nu{}_\sigma = \eta_{\sigma\rho}$. Pauli's fundamental theorem proves the existence of a spinor matrix $S(\Lambda)$ such that

$$(\gamma')^\mu = \Lambda^\mu{}_\nu \gamma^\nu = S(\Lambda)^{-1}\gamma^\mu S(\Lambda)\,. \tag{290}$$

Therefore,

$$0 = \left(i\gamma^\mu \frac{\partial}{\partial x^\mu} - m\right)\psi(x) = S(\Lambda)^{-1}\left(i\gamma^\mu \frac{\partial}{\partial (x')^\nu} - m\right)S(\Lambda)\psi(\Lambda^{-1}x')\,. \tag{291}$$

If we left-multiply with $S(\Lambda)$ and identify $\psi'(x)' = S(\Lambda)\psi(\Lambda^{-1}x')$, we arrive at

$$0 = \left(i\gamma^\mu \frac{\partial}{\partial (x')^\nu} - m\right)\psi'(x')\,, \tag{292}$$

the transformed Dirac equation (287)

To study $S(\Lambda)$ consider an infinitesimal transformation Λ which should also be infinitesimal in S, i.e. we have

$$\Lambda^\mu{}_\nu = \delta^\mu{}_\nu + \omega^\mu{}_\nu\,, \tag{293}$$

$$S(\Lambda) = 1 + i\omega^{\mu\nu}\Sigma_{\mu\nu}\,. \tag{294}$$

Since $\omega_{\mu\nu} + \omega_{\nu\mu} = 0$, we have the same anti-symmetry for Σ. Let us now calculate what happens to γ

$$\Lambda^\mu{}_\nu \gamma^\nu \overset{!}{=} S(\Lambda)^{-1}\gamma^\mu S(\Lambda)\,, \tag{295}$$

$$\Leftrightarrow \quad \gamma^\mu + \omega^\mu{}_\nu \gamma^\nu + \mathcal{O}(\omega^2) = \left(1 - i\omega^{\lambda\rho}\Sigma_{\lambda\rho}\right)\gamma^\mu\left(1 + i\omega^{\lambda\rho}\Sigma_{\lambda\rho}\right) + \mathcal{O}(\omega^2)\,, \tag{296}$$

$$\Leftrightarrow \quad \frac{1}{2}\omega^{\lambda\rho}\left(\delta^\mu{}_\lambda \gamma_\rho - \delta^\mu{}_\rho \gamma_\lambda\right) = \omega^\mu{}_\nu \gamma^\nu = i\omega^{\lambda\rho}\left(\gamma^\mu \Sigma_{\lambda\rho} - \Sigma_{\lambda\rho}\gamma^\mu\right) = i\omega^{\lambda\rho}[\gamma^\mu, \Sigma_{\lambda\rho}] \tag{297}$$

$$\Leftrightarrow \quad \frac{1}{2}\left(\delta^\mu{}_\lambda \gamma_\rho - \delta^\mu{}_\rho \gamma_\lambda\right) = i[\gamma^\mu, \Sigma_{\lambda\rho}]\,. \tag{298}$$

This is satisfied by

$$\Sigma_{\lambda\rho} = -\frac{i}{8}[\gamma_\lambda, \gamma_\rho] \tag{299}$$

which can be integrated to

$$S(\Lambda) = \exp\Big(-\frac{i}{8}\omega^{\mu\nu}[\gamma_\mu, \gamma_\nu]\Big) \qquad \text{for} \quad \Lambda \in \mathrm{SO}^+(1,3)\,. \tag{300}$$

Exercise: Show that

$$[\gamma^\mu, \gamma_\lambda\gamma_\rho] = 2(\delta^\mu{}_\lambda\gamma_\rho - \delta^\mu{}_\rho\gamma_\lambda)\,, \tag{301}$$

by using the anticommutator. Use this to show that (299) is a solution of (298).

The above discussion is only valid for proper transformations that have $\det\Lambda = +1$. To also cover improper transformations, consider

$$P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}\,. \tag{302}$$

This transformation flips the spatial components, i.e. it is a parity transformation. We need to fulfil

$$S(P)^{-1}\gamma^0 S(P) = \gamma^0 \qquad \text{and} \qquad S(P)^{-1}\gamma^i S(P) = -\gamma^i\,. \tag{303}$$

There are two possible choices for $S(P)$

$$S(P) = S(P)^{-1} = \pm\gamma^0 \tag{304}$$

correctly transforms γ^μ. When acting on the wavefunction, we can have two solutions as well

$$\psi(t,\vec{x}) \to \psi'(t,\vec{x}) = \pm\gamma^0\psi(t,-\vec{x})\,. \tag{305}$$

The sign is called intrinsic parity and only starts to matter once we consider system with changing numbers of particles.

5.1.3 Solutions of the Dirac equation

To eventually construct a field theory, we will need a basis of solutions to the wave equation (261). As always, we begin with a Fourier-transformation of $\psi(x)$

$$\psi(x) = \int \frac{\mathrm{d}^3\vec{p}}{(2\pi)^3\sqrt{2E_{\vec{p}}}}\Big(u(p)\mathrm{e}^{-ip\cdot x} + v(p)\mathrm{e}^{ip\cdot x}\Big)\,. \tag{306}$$

Here the u and v objects are vectors in spinor space. They fulfil the momentum-space Dirac equation

$$(\gamma\cdot p - m)u(p) = (\gamma\cdot p + m)v(p) = 0\,. \tag{307}$$

In the restframe of the particle where $p = (E,0,0,0)$, we have

$$(\gamma^0 - 1)u(0) = (\gamma^0 + 1)v(0) = 0\,. \tag{308}$$

We can find explicit answers for the spinors using the explicit representation of (285)

$$u_1(0) = \mathcal{N}\begin{pmatrix}1\\0\\0\\0\end{pmatrix}\,,\quad u_2(0) = \mathcal{N}\begin{pmatrix}0\\1\\0\\0\end{pmatrix}\,,\quad v_1(0) = \mathcal{N}\begin{pmatrix}0\\0\\1\\0\end{pmatrix}\,,\quad v_2(0) = \mathcal{N}\begin{pmatrix}0\\0\\0\\1\end{pmatrix}\,. \tag{309}$$

Note that we have not one but two solution for each direction of p. One can show that these correspond to the two spin directions. At this point the normalisation factor $\mathcal{N}$ is a free parameter. It turns out that $\mathcal{N} = \sqrt{2m}$ but it is possible to modify the following equations to accommodate another normalisation.

To turn these into solutions for $u(p)$, we could perform a Lorentz boost. Alternatively, we can note that

$$(\gamma \cdot p - m)(\gamma \cdot p + m) = (p^2 - m^2)\,, \tag{310}$$

to write

$$u_r(p) \propto (\gamma \cdot p + m)u_r(0)\,, \qquad v_r(p) \propto (-\gamma \cdot p + m)v_r(0)\,. \tag{311}$$

> **Exercise:** (310) relies on the fact that $(\gamma \cdot a)(\gamma \cdot a) = a^2$. Proof this.

A suitable normalisation would be

$$u_r(p) = \frac{\gamma \cdot p + m}{\sqrt{2m(m + E_{\vec{p}})}} u_r(0)\,, \qquad v_r(p) = \frac{-\gamma \cdot p + m}{\sqrt{2m(m + E_{\vec{p}})}} v_r(0)\,, \tag{312}$$

since it leads to be the "boost operator" to be normalised, i.e. for $\vec{p} \to 0$, the fraction has components of one. This choice results in the

$$u_r(p)^\dagger \gamma^0 u_s(p) = 2m\delta_{rs}\,, \qquad v_r(p)^\dagger \gamma^0 v_s(p) = -2m\delta_{rs}\,, \qquad u_r(p)^\dagger \gamma^0 v_s(p) = v_s(p)^\dagger \gamma^0 u_r(p) = 0\,. \tag{313}$$

Further, we can show that (completeness relation)

$$\sum_{r=1,2} u_r(p) u_r(p)^\dagger \gamma^0 = \gamma \cdot p + m \qquad \text{and} \qquad \sum_{r=1,2} v_r(p) v_r(p)^\dagger \gamma^0 = \gamma \cdot p - m\,. \tag{314}$$

5.1.4 Quantisation of the free Dirac field

To define the Lagrangian of the free Dirac field, it is helpful to first define the adjoint spinor

$$\bar{\psi} = \psi^\dagger \gamma^0\,. \tag{315}$$

With this, the Lagrangian can be written as

$$\mathcal{L} = \bar{\psi}(i\gamma^\mu \partial_\mu - m)\psi\,, \tag{316}$$

since it results in the correct Euler-Lagrange equation for $\bar{\psi}$. To see this, we calculate

$$\frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi)} = i\bar{\psi}\gamma^\mu \quad \text{and} \quad \frac{\partial \mathcal{L}}{\partial \psi} = -\bar{\psi} m\,, \tag{317}$$

and write

$$0 = -i\bar{\psi}\overleftarrow{\partial_\mu}\gamma^\mu - m\bar{\psi}\,. \tag{318}$$

Here we have used the notation $\overleftarrow{\partial_\mu}$ to indicate that the derivative is acting to the left rather than the right. We can Hermitian-conjugate this to arrive at (261).

The assosciated Hamiltonian is

$$H = \int \mathrm{d}^3x\; \mathcal{H} = \int \mathrm{d}^3x\; \bar{\psi}(-i\vec{\gamma} \cdot \vec{\nabla} + m)\psi\,. \tag{319}$$

Since u and v are vectors, we will extend our Fourier-decomposition of the fermion field slightly to split the creation and annihilation operators from the spinor vectors, i.e. we write

$$\psi(x) = \int \frac{\mathrm{d}^3\vec{p}}{(2\pi)^3 \sqrt{2E_{\vec{p}}}} \sum_{r=1,2} \Big(a_r(\vec{p}) u_r(p) \mathrm{e}^{-ip\cdot x} + b_r(\vec{p}) v_r(p) \mathrm{e}^{+ip\cdot x} \Big)\,. \tag{320}$$

We have further flipped the propagation direction of the v spinors to ensure positive energy. Recall how we used CCRs to quantise the KG field

$$[\hat{a}(\vec{p}), \hat{a}^\dagger(\vec{q})] = (2\pi)^3 \delta^{(3)}(\vec{p} - \vec{q}) \,, \tag{83a}$$

$$[\hat{a}(\vec{p}), \hat{a}(\vec{q})] = [\hat{a}^\dagger(\vec{p}), \hat{a}^\dagger(\vec{q})] = 0 \,. \tag{83b}$$

Here we would have

$$[a_r(\vec{p}), a_s^\dagger(\vec{q})] = [b_r(\vec{p}), b_s^\dagger(\vec{q})] = (2\pi)^3 \delta^{(3)}(\vec{p} - \vec{q}) \delta_{rs} \qquad \text{and all other commutators zero} \,. \tag{321}$$

However, this will lead to a contradiction. We can show that the Hamiltonian of this theory is

$$H = \int \frac{\mathrm{d}^3 p}{(2\pi)^3} E_{\vec{p}} \sum_r \Big(a_r^\dagger(\vec{p}) a_r(\vec{p}) - b_r^\dagger(\vec{p}) b_r(\vec{p}) \Big) \,. \tag{322}$$

Since we can view $a_r^\dagger a_r$ and $b_r^\dagger b_r$ as particle numbers for particles of type a and b, this would mean that creating more b-type particles decreases the energy of the system.

If we instead swap b and $b^\dagger$

$$\psi(x) = \int \frac{\mathrm{d}^3 \vec{p}}{(2\pi)^3 \sqrt{2E_{\vec{p}}}} \sum_{r=1,2} \Big(a_r(\vec{p}) u_r(p) \mathrm{e}^{-ip\cdot x} + b_r^\dagger(\vec{p}) v_r(p) \mathrm{e}^{+ip\cdot x} \Big) \,, \tag{323}$$

$$\bar{\psi}(x) = \int \frac{\mathrm{d}^3 \vec{p}}{(2\pi)^3 \sqrt{2E_{\vec{p}}}} \sum_{r=1,2} \Big(a_r^\dagger(\vec{p}) \bar{u}_r(p) \mathrm{e}^{+ip\cdot x} + b_r(\vec{p}) \bar{v}_r(p) \mathrm{e}^{-ip\cdot x} \Big) \,, \tag{324}$$

and chose anitcommutation relations

$$\{a_r(\vec{p}), a_s^\dagger(\vec{q})\} = \{b_r(\vec{p}), b_s^\dagger(\vec{q})\} = (2\pi)^3 \delta^{(3)}(\vec{p} - \vec{q}) \delta_{rs} \qquad \text{and all other anticommutators zero} \,, \tag{325}$$

we would find

$$H = \int \frac{\mathrm{d}^3 p}{(2\pi)^3} E_{\vec{p}} \sum_r \Big(a_r^\dagger(\vec{p}) a_r(\vec{q}) + b_r^\dagger(\vec{p}) b_r(\vec{p}) \Big) \,. \tag{326}$$

which means that both $a^\dagger$ and $b^\dagger$ create a particle of mass $m^2 = p^2$.

Exercise: Show (322) and (326).

Exercise: You may find the following quantum mechanics problem instructive. We normally consider the bosonic harmonic oscillator defined as

$$H_B = \frac{\omega}{2}(a^\dagger a + a a^\dagger) \qquad \text{with} \qquad [a, a^\dagger] = 1 \,, \quad [a, a] = [a^\dagger, a^\dagger] = 0 \,. \tag{327}$$

Now define the fermionic oscillator with

$$H_F = \frac{\omega}{2}(b^\dagger b - b b^\dagger) \qquad \text{with} \qquad \{b, b^\dagger\} = 1 \,, \quad \{b, b\} = \{b^\dagger, b^\dagger\} = 0 \,. \tag{328}$$

Write H_F and H_B in terms of number operators $N_F = b^\dagger b$ and $N_B = a^\dagger a$. What are the allowed eigenvalues of N_F and N_B?

You can also define a combined system $H = H_B + H_F$ with $|n\rangle = |n_B\rangle \otimes |n_F\rangle \equiv |n_B, n_F\rangle$. This system treats bosons and fermions the same and is therefore supersymmetric. Show that the supercharge operator $Q = a b^\dagger$ fulfils

$$\{Q, Q\} = \{Q^\dagger, Q^\dagger\} = 0 \,, \quad \omega\{Q, Q^\dagger\} = H \,, \quad [Q, H] = [Q^\dagger, H] = 0 \,. \tag{329}$$

Therefore, Q is a conserved quantity. Finally, apply $Q|n_B, n_F\rangle$ and explain what the operator does to a fermion or boson.

We can now define a few states. As before, we define $|0\rangle$ as the state destroyed by a and b

$$a_r(\vec{p})|0\rangle = b_r(\vec{p})|0\rangle = 0\,. \tag{330}$$

We can also define two different one-particles states

$$|\vec{p},s,+\rangle = \sqrt{2E_{\vec{p}}}a_s^\dagger(\vec{p})|0\rangle \qquad \text{and} \qquad |\vec{p},s,-\rangle = \sqrt{2E_{\vec{p}}}b_s^\dagger(\vec{p})|0\rangle\,. \tag{331}$$

These states are properly normalised such that

$$\langle\vec{p},s,\pm|\vec{q},r,\pm\rangle = 2E_{\vec{p}}\,(2\pi)^3\delta^{(3)}(\vec{p}-\vec{q})\delta_{rs}\,. \tag{332}$$

5.1.5 Charge of the Dirac field and bilinear forms

The Dirac Lagrangian (316) has a symmetry $\psi \to \mathrm{e}^{i\alpha}\psi$ which means that there must be conserved current. This current is

$$j_V^\mu = \bar\psi(x)\gamma^\mu\psi(x)\,. \tag{333a}$$

It is customary to also define

$$j_S = \bar\psi(x)\psi(x)\,, \tag{333b}$$
$$j_5 = \bar\psi(x)\gamma^5\psi(x)\,, \tag{333c}$$
$$j_{5V}^\mu = \bar\psi(x)\gamma^5\gamma^\mu\psi(x)\,. \tag{333d}$$

It is easy to see that j_V^μ and j_{5V}^μ are conserved

$$\partial_\mu j_V^\mu = \big(\partial_\mu\bar\psi\big)\gamma^\mu\psi + \bar\psi\gamma^\mu\big(\partial_\mu\psi\big) = (im\bar\psi)\psi + \bar\psi(-im\psi) = 0\,, \tag{334}$$

Exercise: Show that j_{5V} is conserved as well as long as $m=0$.

Let us also see how j transforms under Lorentz transformation. For example,

$$j_S \to \bar\psi'(x')\psi'(x') = \bar\psi(x)S(\Lambda)^{-1}S(\Lambda)\psi(x) = \bar\psi(x)\psi(x) = j_S(x)\,. \tag{335}$$

Exercise: Use the proof to show that

$$j_V^\mu(x) \to (j_V')^\mu(x') = \Lambda^\mu{}_\nu j_V^\nu(x)\,, \tag{336}$$

and similarly for j_5 and j_{5V}^μ.

The effect of parity is slightly more interesting. For example,

$$j_S \to \bar\psi'(x')\psi'(x') = \bar\psi(x)S(P)^{-1}S(P)\psi(x) = \bar\psi(x)\psi(x) = j_S\,, \tag{337}$$
$$j_5 \to \bar\psi'(x')(\gamma')^5\psi'(x') = \bar\psi(x)S(P)^{-1}\gamma^5S(P)\psi(x) = -\bar\psi(x)\gamma^5\psi(x) = -j_5\,, \tag{338}$$

and similarly for the vector currents. We have used that

$$S(P)^{-1}\gamma^5S(P) = \gamma^0\gamma^5\gamma^0 = -\gamma^0\gamma^0\gamma^5 = -\gamma^5\,. \tag{339}$$

This is the original of the labels we have used for the different basis elements in (283): since j_S (j_V) transforms like a Lorentz scalar (Lorentz vector) we call it a scalar (vector) current. The "pseudo" prefix indicates that the current picks up a sign under parity conservation, the same way that e.g. the angular momentum $\vec{L} = \vec{x}\times\vec{p}$ does.

Classically, the vector current j_V corresponds to the electromagnetic current with the charge density as $\rho = j_V^0$. Let us calculate this current for our QFT

$$Q = \int \mathrm{d}^3x \ \rho(x) = \int \mathrm{d}^3x \ \psi^\dagger(x)\psi(x) \tag{340}$$

$$= \int \frac{\mathrm{d}^3x \ \mathrm{d}^3\vec{q} \ \mathrm{d}^3\vec{p}}{(2\pi)^6\sqrt{2E_{\vec{p}}2E_{\vec{q}}}} \sum_{r,s} \Big(a_r^\dagger(\vec{p})u_r(p)^\dagger \mathrm{e}^{+ip\cdot x} + b_r(\vec{p})v_r(p)^\dagger \mathrm{e}^{-ip\cdot x} \Big)\Big(a_r(\vec{p})u_r(p)\mathrm{e}^{-ip\cdot x} + b_r^\dagger(\vec{p})v_r(p)\mathrm{e}^{+ip\cdot x} \Big) \tag{341}$$

$$= \int \frac{\mathrm{d}^3\vec{p}}{(2\pi)^3} \sum_r \Big(a_r^\dagger(\vec{p})a_r(\vec{p}) + b_r(\vec{p})b_r^\dagger(\vec{p}) \Big) = \int \frac{\mathrm{d}^3\vec{p}}{(2\pi)^3} \sum_r \Big(a_r^\dagger(\vec{p})a_r(\vec{p}) - b_r^\dagger(\vec{p})b_r(\vec{p}) \Big) , \tag{342}$$

where we have dropped yet another $\int \mathrm{d}^3p 1$ constant in the last step. This proves that the particles created by $a^\dagger$ have charge $Q = +1$ and the those created by $b^\dagger$ have $Q = -1$ Therefore, we call the former particles and the latter antiparticles.

Exercise: Show the above.

5.1.6 Dirac propagator

Finally, we should calculate the propagator of a fermion. To do this, we simply write

$$\langle 0|\psi_a(x)\bar{\psi}_b(y)|0\rangle = \int \frac{\mathrm{d}^3p}{(2\pi)^3 2E_{\vec{p}}} \mathrm{e}^{ip\cdot(x-y)} \underbrace{\sum_r u_r(p)_a \bar{u}_r(p)_b}_{(\gamma\cdot p+m)_{ab}} , \tag{343}$$

$$\langle 0|\bar{\psi}_b(y)\psi_a(x)|0\rangle = \int \frac{\mathrm{d}^3p}{(2\pi)^3 2E_{\vec{p}}} \mathrm{e}^{ip\cdot(y-y)} \underbrace{\sum_r v_r(p)_a \bar{v}_r(p)_b}_{(\gamma\cdot p-m)_{ab}} . \tag{344}$$

Keep in mind that ψ and $\bar{\psi}$ are operator-valued vector fields and therefore have spinor indices a and b, not to be confused with the *operators* a and b. By writing $p_\mu \to i\partial_\mu$, we can pull the Dirac structure out and are left with $D(x-y)$ of (116)

$$\langle 0|\psi_a(x)\bar{\psi}_b(y)|0\rangle = +(i\gamma\cdot\partial_x + m)_{ab}D(x-y) , \tag{345}$$

$$\langle 0|\bar{\psi}_b(y)\psi_a(x)|0\rangle = -(i\gamma\cdot\partial_x - m)_{ab}D(y-x) . \tag{346}$$

Similarly, the delayed Green's function can be written as

$$S_R^{ab}(x-y) = \theta(x^0-y^0)\langle 0|\{\psi_a(x), \bar{\psi}_b(y)\}|0\rangle = (i\gamma\cdot\partial_x + m)_{ab}D_R(x-y) . \tag{347}$$

Exercise: Verify that S_R is indeed a Green's function of the Dirac operator $i\partial - m$.

For the Fourier-transformed Green's function, we find

$$\tilde{S}_R(x-y) = \frac{i(\gamma\cdot p + m)}{p^2 - m^2} = \frac{i}{\gamma\cdot p - m} . \tag{348}$$

We can use the same constructions to define the Feynman propagator

$$S_F(x-y) = \int \frac{\mathrm{d}^4p}{(2\pi)^4} \frac{i}{\gamma\cdot p - m + i\epsilon} \mathrm{e}^{-ip\cdot(x-y)} = \langle 0|T\psi(x)\bar{\psi}(y)|0\rangle . \tag{349}$$

5.2 Vector fields

The last particle we will consider is the photon, i.e. the particle of the electromagnetic field. The classical Lagrangian is just

$$\mathcal{L} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} \quad \text{with} \quad F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu \,. \tag{350}$$

$F^{\mu\nu}$ is called the field-strength tensor and A^μ the vector potential. We will not discuss how to quantise this field because A is a gauge field which makes canonical quantisation much more complicated. This is because the conjugate momentum to A^0 is zero as $\mathcal{L}$ does not contain $\dot{A}^0$ (the term $\partial^0 A^0$ would be in $F^{00} = \partial^0 A^0 - \partial^0 A^0 = 0$). The way to circumvent this problem is add a gauge-fixing term to $\mathcal{L}$ that forces a specific gauge, e.g. Lorentz gauge, i.e. $\partial_\mu A^\mu = 0$, in which we can write down CCRs

$$[A^\mu(x), \dot{A}^\nu(y)] = -i\eta^{\mu\nu}\delta(x-y)\,. \tag{351}$$

This is very similar the KG field and we can almost proceed along the same lines[8]. The photon field can be written as

$$A^\mu(x) = \int \frac{\mathrm{d}^3 p}{(2\pi)^3\sqrt{2E_{\vec{p}}}} \sum_\lambda \epsilon^\mu_\lambda(p)\left[a_{p,\lambda}\mathrm{e}^{ip\cdot x} + a^\dagger_{p,\lambda}\mathrm{e}^{-ip\cdot x}\right]. \tag{352}$$

Here we use λ to sum over polarisations and ϵ to denote the polarisation vector itself. Naively we would expect two polarisations. However, the gauge fixing leads to two unphysical polarisations that we also need to sum over. Like the u and v spinors, the polarisation vector ϵ has a complenetess relation

$$\sum_\lambda \epsilon^\mu_\lambda(p)\epsilon^{*\nu}_\lambda(p) = -\eta^{\mu\nu}\,. \tag{353}$$

The Feynman propagator of this theory is

$$D_F^{\mu\nu}(x-y) = \int \frac{\mathrm{d}^4 p}{(2\pi)^4}\frac{\eta^{\mu\nu}}{p^2+i\epsilon}\mathrm{e}^{-ip\cdot(x-y)}\,. \tag{354}$$

[8]There is one more problem related to the fact that states created by $a^{0\dagger}$ have negative norm.

6 Quantum electrodynamics

We can finally write down and work with the QED Lagrangian. We want our theory of electrodynamics to include electrons and photons, i.e. our free Lagrangian will be the sum of (316) and (316)

$$\mathcal{L}_0 = \bar{\psi}(i\gamma \cdot \partial - m)\psi - \frac{1}{4}F^{\mu\nu}F_{\mu\nu}\,. \tag{355}$$

To achieve interactions between the two types of particles, recall the concept of minimal coupling: in classical field theory one couples a particle to the electromagnetic field by shifting its momentum $p \to p - qA$. The quantum equivalent of this is shift

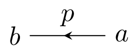
$$i\partial_\mu \to i\partial_\mu - qA_\mu \equiv iD_\mu\,. \tag{356}$$

The object D_μ is usually called the gauge-covariant derivative for reasons we will see shortly. The means the QED Lagrangian can be written as

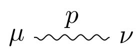
$$\mathcal{L} = \bar{\psi}(i\gamma \cdot D - m)\psi - \frac{1}{4}F^{\mu\nu}F_{\mu\nu} = \mathcal{L}_0 - e\bar{\psi}\gamma^\mu\psi A_\mu\,. \tag{357}$$

The simplicity of this results is one of the most remarkable features of modern physics. It accurately describes nature except for gravity and nuclear phenomena which require the strong and weak nuclear forces. However, both of these have the same structure (albeit with a different gauge group) and their fields can be absorbed into F and D.

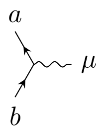
We can write down Feynman rules for this theory

For each internal fermion 

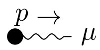
$$= \left(\frac{i}{\gamma \cdot p - m + i\epsilon}\right)_{ab}, \tag{358}$$

For each internal photon 

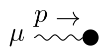
$$= \frac{-i\eta_{\mu\nu}}{p^2 + i\epsilon}, \tag{359}$$

For each vertex 

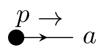
$$= -ie\gamma^\mu_{ab}, \tag{360}$$

For each external photon 

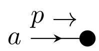
$$= \epsilon^*_\mu(p) \quad \text{(final)}, \tag{361}$$

For each external photon 

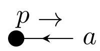
$$= \epsilon_\mu(p) \quad \text{(initial)}, \tag{362}$$

For each external fermion 

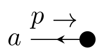
$$= \big(\bar{u}_s(p)\big)_a \quad \text{(final)}, \tag{363}$$

For each external fermion 

$$= \big(u_s(p)\big)_a \quad \text{(initial)}, \tag{364}$$

For each external antifermion 

$$= \big(v_s(p)\big)_a \quad \text{(final)}, \tag{365}$$

For each external antifermion 

$$= \big(\bar{v}_s(p)\big)_a \quad \text{(initial)}\,. \tag{366}$$

We use the black blob to indicate the rest of the process and the direction of the momentum arrow relative to it to indicate incoming or outgoing particles.

We have used the spinors u and v to denote external fermion fields and the polarisation vector ϵ to denote external photons. For fermions, we have to keep track of *both the flow of momentum and the spinor flow* which is what is indicated by the arrow on the line. For the propagator, *we have assumed they are aligned*, otherwise we pick up a relative sign between $\gamma \cdot p$ and m.

Further note that we will drop the spinor indices in the future and often just write e.g.

$$\bar{u}(p)\gamma^\mu v(q) \equiv \big[\bar{u}_{s_p}(p)\big]_a \big[\gamma^\mu\big]_{a,b} \big[v_{s_q}(q)\big]_b\,. \tag{367}$$

6.1 Gauge structure

Note that (357) has a gauge symmetry

$$\psi(x) \to \mathrm{e}^{i\alpha(x)}\psi(x) \quad \text{while} \quad A_\mu(x) \to A_\mu - \frac{1}{q}\partial_\mu \alpha(x)\,. \tag{368}$$

The gauge-covariant derivative transforms like the field under this symmetry

$$D_\mu \psi(x) \to \Big(\partial_\mu + ieA_\mu - i(\partial_\mu \alpha)\Big)\mathrm{e}^{i\alpha}\psi(x) = \mathrm{e}^{i\alpha}\Big(\partial_\mu + ieA_\mu\Big)\psi(x) = \mathrm{e}^{i\alpha} D_\mu \psi(x)\,, \tag{369}$$

leaving $\mathcal{L}$ invariant. This means it would have been possible to *state* $\mathcal{L}$ based on the requirement that the gauge symmetry holds without thinking about minimal coupling at all!

The photon as a consequence of local symmetry
In fact, we can go further and derive the complete Lagrangian, including the photon field, from the fact that

$$\mathcal{L} = \bar{\psi}(i\gamma \cdot \partial - m)\psi \tag{370}$$

should be invariant under local gauge transformation

$$\psi(x) \to \mathrm{e}^{i\alpha(x)}\psi(x)\,. \tag{371}$$

The mass term is obviously invariant but what about the derivative? Due to the gauge symmetry, the derivative ∂_μ no longer has any geometric meaning since the phase $\alpha(x)$ could mess things up. Let us therefore define a new derivate D_μ which compares two nearby points along a direction $\hat{n}^\mu$

$$\hat{n}^\mu D_\mu \psi = \lim_{\epsilon\to 0} \frac{\psi(x+\epsilon\hat{n}) - U(x+\epsilon\hat{n}, x)\psi(x)}{\epsilon}\,. \tag{372}$$

Here we had to introduce a new object, U, which for the normal derivative ∂_μ is just $U = 1$ but accounts for the change α. For $\mathcal{L}$ to be invariant, we need this new derivative to transform like the field itself, i.e.

$$\hat{n}^\mu D_\mu \psi \to \lim_{\epsilon\to 0} \frac{\mathrm{e}^{i\alpha(x+\epsilon\hat{n})}\psi(x+\epsilon\hat{n}) - U'(x+\epsilon\hat{n}, x)\mathrm{e}^{i\alpha(x)}\psi(x)}{\epsilon} \overset{!}{=} \mathrm{e}^{i\alpha(x)}\hat{n}^\mu D_\mu \psi\,. \tag{373}$$

The only way to more this work generally is if U transforms as

$$U(y,x) \to U'(y,x) = \mathrm{e}^{i\alpha(y)}U(y,x)\mathrm{e}^{-i\alpha(x)}\,. \tag{374}$$

Then we have

$$\hat{n}^\mu D_\mu \psi \to \lim_{\epsilon\to 0} \mathrm{e}^{i\alpha(x+\epsilon\hat{n})} \frac{\psi(x+\epsilon\hat{n}) - U(x+\epsilon\hat{n}, x)\psi(x)}{\epsilon} = \mathrm{e}^{i\alpha(x)}\hat{n}^\mu D_\mu \psi\,. \tag{375}$$

Taylor-expanding U gives us with $U(x,x) = 1$

$$U(x+\epsilon\hat{n}, x) = 1 - i\epsilon\hat{n}^\mu(eA_\mu(x)) + \mathcal{O}(\epsilon^2)\,. \tag{376}$$

Here we had to introduce a field A_μ that is the derivative of U as well as an arbitrary constant e. It is easy to see that A_μ transforms as required and it is no surprise that it will turn into the photon field.

We can now concatenate four comparison operations into a small square

$$\mathcal{U}(x) = U(x, x+\epsilon\hat{n})U(x+\epsilon\hat{n}, x+\epsilon\hat{n}+\epsilon\hat{m})U(x+\epsilon\hat{n}+\epsilon\hat{m}, x+\epsilon\hat{m})U(x+\epsilon\hat{m}, x)\,. \tag{377}$$

It is easy to see that $\mathcal{U}(x)$ is invariant under the transformation. Starting from

$$U(x,y) = \exp\Big(-ieA\big(\frac{x+y}{2}\big)\cdot(x-y) + \mathcal{O}\big((x-y)^3\big)\Big)\,, \tag{378}$$

we find

$$
\begin{aligned}
\mathcal{U}(x) &= \exp\Big[-ie\epsilon\Big(-\hat{n}\cdot A\big(x+\hat{n}\tfrac{\epsilon}{2}\big)-\hat{m}\cdot A\big(x+\hat{n}\epsilon+\hat{m}\tfrac{\epsilon}{2}\big)+\hat{n}\cdot A\big(x+\hat{m}\epsilon+\hat{n}\tfrac{\epsilon}{2}\big)+\hat{m}\cdot A\big(x+\hat{m}\tfrac{\epsilon}{2}\big)\Big)\Big] \\
&= \exp\Big[-ie\epsilon\big(\partial_{\hat{m}}(\hat{n}\cdot A)-\partial_{\hat{n}}(\hat{m}\cdot A)\big)\Big)\,. \qquad (379)
\end{aligned}
$$

This proofs that $F_{\mu\nu}$ and any functions that depend on $F_{\mu\nu}$ are invariant. However, A_μ itself is not invariant meaning that a mass term like $m_\gamma\ A_\mu A^\mu$ would not be allowed. This is the reason that the photon is massless.

You may now wonder about the Z and W bosons. The same argument still applies and we cannot write down a mass for them. In the Standard Model, their masses are dynamically generated through the Higgs mechanism. Basically, the theory contains a scalar field ϕ whose kinetic term includes a covariant deriviative that couples it dynamically to the W and Z bosons. Uniquely among all particles, this field has a non-zero vev meaning that W and Z get a *dynamically* generated mass.

6.2 Recipe for evaluations

The following is a rough recipe for evaluating Feynman diagrams. We will shortly discuss each step in an example but it may be convenient to have it all in one place. To calculate an amplitude,

1. draw all the Feynman diagrams.
2. assign momenta to all edges and Lorentz indices to all photon vertices.
3. pick the end of any fermion line and follow the arrow backwards, evaluating as you go.
4. multiply in the polarisation tensors for any external photons and propagators for each internal photon.
5. contract any open index
6. if your diagram has any loops, integrate over the unconstrained momenta. If you have any internal fermion loops, calculate the trace over their gamma matrices.
7. you now have the amplitude $\mathcal{M}$.

To calculate a cross section,

8. square the amplitude as $|\mathcal{M}|^2 = \mathcal{M}^\dagger \times \mathcal{M}$. Make sure to rename any indices to avoid collisions. To calculate $\mathcal{M}^\dagger$, you can use the fact that $(A\cdot B)^\dagger = B^\dagger A^\dagger$ and

   $$u^\dagger = u^\dagger\gamma^0\gamma^0 = \bar{u}\gamma^0 \qquad \text{and} \qquad \bar{u}^\dagger = \big[u^\dagger\gamma^0\big]^\dagger = \big[\gamma^0\big]^\dagger u = \gamma^0 u\,, \qquad (380)$$

   and similarly for v. Further,

   $$(\gamma^\mu)^\dagger = \gamma^0\gamma^\mu\gamma^0\,. \qquad (275)$$

   As we will see, this just amounts to reversing the spin line.

9. if you calculate unpolarised scattering, sum over final-state and average over initial-state polarisations. For this, the completeness relations (314) and (353) will be helpful

   $$
   \begin{aligned}
   \sum_{r=1,2} u_r(p)\bar{u}_r(p) &= \gamma\cdot p + m\,, \\
   \sum_{r=1,2} v_r(p)\bar{v}_r(p) &= \gamma\cdot p - m\,, \qquad (314) \\
   \sum_{\lambda} \epsilon_{\lambda,\mu}(p)\epsilon^*_{\lambda,\nu}(p) &= -\eta_{\mu\nu}\,. \qquad (353)
   \end{aligned}
   $$

10. for fermions, this will result in a trace of γ matrices (see below). Evaluate this trace using trace identities (271)

$$\begin{aligned}
\mathrm{tr}(\gamma^\mu) &= 0\,, \\
\mathrm{tr}(\underbrace{\gamma^{\mu_1}\cdots\gamma^{\mu_k}}_{\text{odd}}) &= 0\,, \\
\mathrm{tr}(\gamma^\mu\gamma^\nu) &= 4\eta^{\mu\nu}\,, \\
\mathrm{tr}\Big(\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\Big) &= 4\Big(\eta^{\mu\nu}\eta^{\rho\sigma} - \eta^{\mu\rho}\eta^{\nu\sigma} + \eta^{\mu\sigma}\eta^{\nu\rho}\Big)\,.
\end{aligned} \tag{271}$$

11. use (232) or (234) to relate this to the cross section or decay rate

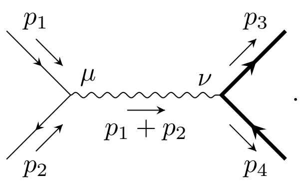
$$\mathrm{d}\sigma = \frac{1}{(2E_a)(2E_b)|\vec{v}|}\underbrace{\left(\prod_{n=1}^{f}\frac{\mathrm{d}^3p_n}{(2\pi)^3 2E_n}\right)(2\pi)^4\delta^{(4)}\left(p_a + p_b - \sum_{n=1}^{f}p_n\right)}_{\mathrm{d}\Phi_{2\to f}}\ |\mathcal{M}|^2\,, \tag{232}$$

$$\mathrm{d}\Gamma = \frac{1}{2M}\underbrace{\left(\prod_{n=1}^{f}\frac{\mathrm{d}^3p_n}{(2\pi)^3 2E_n}\right)(2\pi)^4\delta^{(4)}\left(p_a - \sum_{n=1}^{f}p_n\right)}_{\mathrm{d}\Phi_{1\to f}}\ |\mathcal{M}|^2\,. \tag{234}$$

12. integrate over phase space.

6.3 A simple process $ee \to \mu\mu$

Let us calculate our first real process, $ee \to \mu\mu$. There is just a single s-channel diagram contributing so Step 1 and 2 are very simple

$\mathcal{M} =$  . (381)

We have to apply Step 3 twice, once for the muon and once for the electron. After Step 5, we have

$$\mathcal{M} = \Big[\bar{v}_{s_2}(p_2)\big(ie\gamma^\mu\big)u_{s_1}(p_1)\Big]\frac{-i\eta_{\mu\nu}}{(p_1+p_2)^2 + i\epsilon}\Big[\bar{u}_{s_3}(p_3)\big(ie\gamma^\nu\big)v_{s_4}(p_4)\Big] \tag{382}$$

$$= \frac{ie^2}{(p_1+p_2)^2 + i\epsilon}\Big[\bar{v}_{s_2}(p_2)\gamma^\mu u_{s_1}(p_1)\Big]\Big[\bar{u}_{s_3}(p_3)\gamma_\mu v_{s_4}(p_4)\Big]\,. \tag{383}$$

We can now square this object

$$|\mathcal{M}|^2 = \frac{e^4}{\big[(p_1+p_2)^2\big]^2}\Big[\bar{v}_{s_2}(p_2)\gamma^\mu u_{s_1}(p_1)\ \bar{u}_{s_3}(p_3)\gamma_\mu v_{s_4}(p_4)\Big]^\dagger\Big[\bar{v}_{s_2}(p_2)\gamma^\nu u_{s_1}(p_1)\ \bar{u}_{s_3}(p_3)\gamma_\nu v_{s_4}(p_4)\Big]\,. \tag{384}$$

Let us first work on the first bracket $[\cdots]^\dagger$ and use that

$$\Big[\cdots\Big]^\dagger = \big[u_{s_1}(p_1)\big]^\dagger\big[\gamma^\mu\big]^\dagger\big[\bar{v}_{s_2}(p_2)\big]^\dagger\ \big[v_{s_4}(p_4)\big]^\dagger\big[\gamma_\mu\big]^\dagger\big[\bar{u}_{s_3}(p_3)\big]^\dagger \tag{385}$$

$$= \bar{u}_{s_1}(p_1)\gamma^0\ \gamma^0\gamma^\mu\gamma^0\ \gamma^0 v_{s_2}(p_2)\ \bar{v}_{s_4}(p_4)\gamma^0\ \gamma^0\gamma_\mu\gamma^0\ \gamma^0 u_{s_3}(p_3) \tag{386}$$

$$= \bar{u}_{s_1}(p_1)\gamma^\mu v_{s_2}(p_2)\ \bar{v}_{s_4}(p_4)\gamma_\mu u_{s_3}(p_3)\,. \tag{387}$$

This means we now have after re-bracketing things

$$|\mathcal{M}|^2 = \frac{e^4}{\big[(p_1+p_2)^2\big]^2}\Big[\bar{u}_{s_1}(p_1)\gamma^\mu v_{s_2}(p_2)\bar{v}_{s_2}(p_2)\gamma^\nu u_{s_1}(p_1)\Big]\Big[\bar{v}_{s_4}(p_4)\gamma_\mu u_{s_3}(p_3)\bar{u}_{s_3}(p_3)\gamma_\nu v_{s_4}(p_4)\Big]\,. \tag{388}$$

Let us look at one of those, e.g. the first one where we have $v_{s_2}\bar{v}_{s_2}$. If we sum over the polarisation states, i.e. calculate $v_1\bar{v}_1 + v_2\bar{v}_2$, we can write this using the completeness relation as $\gamma \cdot p_2 - m$.

$$\sum_{s_1,s_2}\Big[\cdots\Big] = \sum_{s_1}\bar{u}_{s_1}(p_1)\gamma^\mu\big(\gamma\cdot p_2 - m\big)\gamma^\nu u_{s_1}(p_1) \tag{389}$$

To make use of this also for s_1, remember that the bracket is a number under spinor, i.e.

$$\bar{u}(p_1)\cdots u(p_1) = \mathrm{tr}\Big[\bar{u}(p_1)\cdots u(p_1)\Big] = \mathrm{tr}\Big[u(p_1)\bar{u}(p_1)\cdots\Big] = \mathrm{tr}\Big[\big(\gamma\cdot p_1 + m\big)\cdots\Big]\,. \tag{390}$$

This is sometimes referred to the Casimir trick and it is essential to calculating matrix elements with traces. Therefore,

$$\sum_{s_1,s_2}\Big[\cdots\Big] = \mathrm{tr}\Big[\big(\gamma\cdot p_1 + m\big)\gamma^\mu\big(\gamma\cdot p_2 - m\big)\gamma^\nu\Big]\,. \tag{391}$$

We now can expand and calculate this trace

$$\sum_{s_1,s_2}\Big[\cdots\Big] = p_{1,\rho}p_{2,\sigma}\mathrm{tr}\Big[\gamma^\rho\gamma^\mu\gamma^\sigma\gamma^\nu\Big] - m\;\mathrm{tr}\Big[\gamma\cdot p_1\;\gamma^\mu\;\gamma^\nu\Big] + m\;\mathrm{tr}\Big[\gamma^\mu\;\gamma\cdot p_2\;\gamma^\nu\Big] - m^2\mathrm{tr}\Big[\gamma^\mu\gamma^\nu\Big] \tag{392}$$

$$\begin{aligned} &= p_{1,\rho}p_{2,\sigma}4\Big(\eta^{\rho\mu}\eta^{\sigma\nu} - \eta^{\rho\sigma}\eta^{\mu\nu} + \eta^{\rho\nu}\eta^{\mu\sigma}\Big) - m^2 4\eta^{\mu\nu} \\ &= 4p_1^\mu p_2^\nu + 4p_1^\nu p_2^\mu - 4(p_1\cdot p_2 + m^2)\eta^{\mu\nu} \end{aligned} \tag{393}$$

Doing the same for the other bracket, we have

$$\sum_{s_3,s_4}\Big[\cdots\Big] = 4p_{3,\mu}p_{4,\nu} + 4p_{3,\nu}p_{4,\mu} - 4(p_3\cdot p_4 + M^2)\eta_{\mu\nu}\,. \tag{394}$$

and with $s = (p_1+p_2)^2$ and $t = (p_1 - p_3)^2$

$$\sum_{s_i}|\mathcal{M}|^2 = \frac{16q^4}{\big[(p_1+p_2)^2\big]^2}\Big[p_1^\mu p_2^\nu + p_1^\nu p_2^\mu - (p_1\cdot p_2 + m^2)\eta^{\mu\nu}\Big]\Big[p_{3,\mu}p_{4,\nu} + p_{3,\nu}p_{4,\mu} - (p_3\cdot p_4 + M^2)\eta_{\mu\nu}\Big] \tag{395}$$

$$= \frac{8e^4}{s^2}\Big(2m^4 + 4m^2\left(M^2 - t\right) + 2M^4 - 4M^2 t + s^2 + 2st + 2t^2\Big)\,, \tag{396}$$

where we have used that $\eta^{\mu\nu}\;\eta_{\mu\nu} = 4$.

To simplify the discussion of the cross section, let us set $m = 0$ (high energy limit) and write t in terms of $\cos\theta$ (cf. (255b))

$$t = (p_1 - p_3)^2 = \frac{s}{2}\Big(-\frac{1+\beta^2}{2} + \beta\;\cos\theta\Big)\quad\text{with}\quad \beta = \sqrt{1 - \frac{4M^2}{s}} \tag{397}$$

the velocity of the muon. Now the matrix element squared is

$$\sum_{s_i}|\mathcal{M}|^2 = 4q^4\Big(2 - (1-\cos^2\theta)\beta^2\Big)\,. \tag{398}$$

Since we have to average over 2×2 incoming spins, the four cancels. The cross section is

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{\alpha^2}{4s}\beta\Big(2 - (1-\cos^2\theta)\beta^2\Big)\,. \tag{399}$$

In the high-energy limit, $M = 0$ or $\beta = 1$, this is

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{\alpha^2}{4s}(1+\cos^2\theta)\,. \tag{400}$$

At this point, there are a number of exercises you can do. In order of increasing complexity.

Exercise: Calculate $\mathrm{d}\sigma/\mathrm{d}t$ for $e\mu \to e\mu$ scattering. The amplitude squared is

$$\sum |\mathcal{M}|^2 = \frac{8e^4}{t^2}\Big(2m^4 + 2M^4 + 4m^2(M^2 - s) - 4M^2 s + 2s^2 + 2st + t^2\Big)\,, \tag{401}$$

and the cross section is

$$\frac{\mathrm{d}\sigma}{\mathrm{d}t} = 4\pi\alpha^2 \frac{(M^2+m^2)^2 - su + t^2/2}{t^2\lambda}\,, \tag{402}$$

with $\lambda = m^4 - 2m^2M^2 + M^4 - 2m^2 s - 2M^2 s + s^2$.

Exercise: Calculate the cross section for $ee \to \gamma\gamma$. The amplitude can be written down as

$$\begin{aligned}\mathcal{M} &= \bar{v}(p_2)(-ie\gamma^\mu)\frac{i}{\gamma\cdot(p_1-p_4) - m}(-ie\gamma^\nu)u(p_1)\ \epsilon^*_\mu(p_3)\epsilon^*_\nu(p_4) \\ &+ \bar{v}(p_2)(-ie\gamma^\nu)\frac{i}{\gamma\cdot(p_1-p_3) - m}(-ie\gamma^\mu)u(p_1)\ \epsilon^*_\mu(p_3)\epsilon^*_\nu(p_4)\,.\end{aligned} \tag{403}$$

Exercise: Calculate the cross section $\mathrm{d}\sigma/\mathrm{d}t$ for $e^+e^- \to e^+e^-$. We find

$$\sum |\mathcal{M}|^2 = 8e^4\left(\frac{8m^4 - 8m^2 s + 2s^2 + 2st + t^2}{t^2} + \frac{8m^4 + s^2 - 8m^2 t + 2st + 2t^2}{s^2} + \frac{2(s+t)^2 - 8m^4}{st}\right). \tag{404}$$

Exercise: Write down, but do not calculate, the amplitudes at $\mathcal{O}(e^4)$ for $ee \to \mu\mu$.

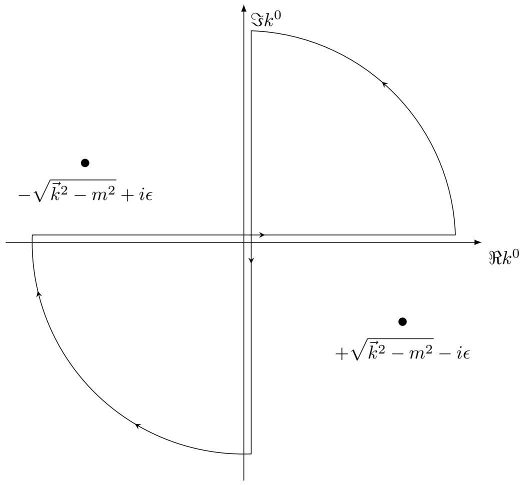
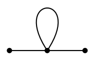


Figure 6: The contour of the Wick rotation

7 Higher orders

So far we have only calculated observables at the first order in perturbation theory. This is usually fine to get a rough idea of the cross section but especially in QCD where $\alpha_s \sim 0.1$, higher-order corrections can be very large (up to 100%). If we want to do any kind of precision measurement, either at the LHC or elsewhere like the anomalous magnetic moments from the abstract, we need more precision. Luckily, we have the tools to do this by simply drawing more complicated Feynman diagrams.

Consider for example the first correction to the ϕ propagator in ϕ^4 (cf. (191))

$\langle\Omega|T\{\phi(x)\phi(y)\}|\Omega\rangle =$ $+$  $+\mathcal{O}(\lambda^2)$ (405)

$$= \frac{1}{p^2 - m^2 + i\epsilon} + \frac{1}{p^2 - m^2 + i\epsilon}\frac{\lambda}{2}\int \frac{\mathrm{d}^4k}{(2\pi)^4}\frac{i}{k^2 - m^2 + i\epsilon}\frac{1}{p^2 - m^2 + i\epsilon} + \mathcal{O}(\lambda^2)\,. \tag{406}$$

Let us focus on the integral. To calculate it, we should first try to replace the integral over the Minkowskian momentum k into a Euclidean one. We currently have $k^2 = (k^0)^2 - \vec{k}^2$. If we could replace $k^0 \to ik^4$, we have instead

$$k^2 = (k^0)^2 - \vec{k}^2 \to -(k^4)^2 - \vec{k}^2 \equiv -k_E^2\,, \tag{407}$$

$$\mathrm{d}^4k = \mathrm{d}k^0\mathrm{d}^3\vec{k} \to i\mathrm{d}^4k_E\,. \tag{408}$$

k_E is now a normal Euclidean vector that integrate normally. By looking at the integration counter shown in Figure 6, we can see that this does not change the integral. The poles of the k^0 integration are at $k^0 = \pm\sqrt{\vec{k}^2 + m^2} \mp i\epsilon$ in the top-left and lower-right quadrant. This means the integral over the whole contour vanishes

$$0 = \oint \mathrm{d}k^0 = \left(\int_{-\infty}^{\infty} + \int_{-i\infty}^{i\infty} + \text{arcs}\right)\,. \tag{409}$$

Assuming the arcs vanish, we can write

$$\int \frac{\mathrm{d}^4k}{(2\pi)^4}\frac{1}{k^2 - m^2 + i\epsilon} = -i\int \frac{\mathrm{d}^4k_E}{(2\pi)^4}\frac{1}{k_E^2 + m^2}\,. \tag{410}$$

This procedure is called Wick rotation.

Before we start calculating, let us just look at the integral. If $k \gg m$, the integral will scale like $k_E^4/k_E^2 \sim k_E^2 \to \infty$, i.e. the integral is divergent! What does this mean for the whole concept of QFT?

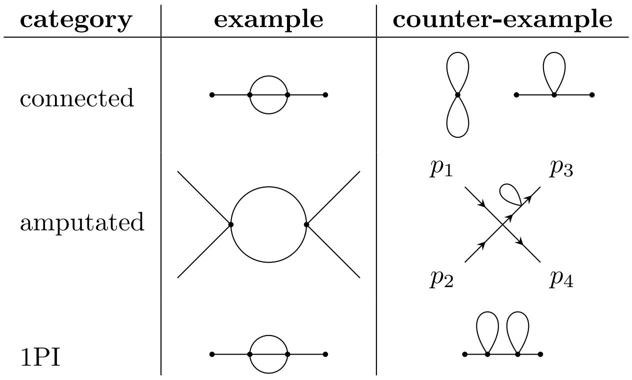


Figure 7: Some example diagrams that are connected or unconnect, amputated or non-amputated, 1PI or non-1PI.

If higher-order terms can diverge, the λ suppression does not really matter and we loose all predictive power.

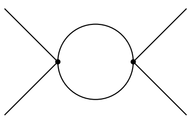
Before we continue, we should introduce one more item of terminology. We already know how to classify by connected vs. disconnected and by amputated vs. non-amputated. Now we introduce one-particle irreducible (1PI) diagrams that cannot by split into two diagrams by cutting a single line. For an example of the three categorisation, see Table 7.

7.1 Regularisation

To fix this problem, we first need to make it manifest. This means we need to introduce some way of parametrising the problem through a process called regularisation that we have encountered before. In the following section we will concurrently develop two regularisation techniques: cut-off regularisation and dimensional regularisation (dimreg). The former is conceptually easier to understand but very difficult to implement in practice. The latter may sound a bit more abstract and esoteric but is how almost all modern calculations are carried out.

Additionally to $\langle\Omega|T\{\phi(x)\phi(y)\}|\Omega\rangle$, we will also calculate the four-point function

$$\langle p_4 p_3|T|p_2 p_1\rangle\Big|_{1L,p=0} =$$



$$+\text{ permutations} = (-i)(-i\lambda)^2\frac{3}{2}\int\frac{\mathrm{d}^4k}{(2\pi)^4}\frac{i^2}{(k^2-m^2+i\epsilon)^2} \tag{411}$$

7.1.1 Cut-off regularisation

Since our problem is due to k being very large, let us just truncate the integral at some large value Λ. We can now simply write

$$\langle\Omega|T\{\phi(x)\phi(y)\}|\Omega\rangle\Big|_{1L} = \frac{-\lambda}{32\pi^4}\int_\Lambda\frac{\mathrm{d}^4k_E}{k_E^2+m^2} = \frac{-\lambda}{16\pi^2}\int_0^\Lambda \mathrm{d}k_E\frac{k_E^3}{k_E^2+m^2} = \frac{-\lambda}{32\pi^2}\left(\Lambda^2 - m^2\log\frac{m^2+\Lambda^2}{m^2}\right)\,, \tag{412}$$

where we have used that

$$\mathrm{d}^4k_E = \mathrm{d}k_E\; k_E^3\;\mathrm{d}\Omega^{(4)} \quad\text{and}\quad \int\mathrm{d}\Omega^{(4)} = 2\pi^2\,. \tag{413}$$

Expanding this in Λ^2/m^2, we find

$$\langle\Omega|T\{\phi(x)\phi(y)\}|\Omega\rangle\Big|_{1L} = -\frac{m^2\lambda}{32\pi^2}\left(\frac{\Lambda^2}{m^2} - \log\frac{\Lambda^2}{m^2}\right) + \mathcal{O}\big((\Lambda/m)^0\big)\,. \tag{414}$$

This does not solve the problem but makes it explicit enough that we can talk about it.

Exercise: Calculate the other process to show that

$$\langle p_4 p_3|T|p_2 p_1\rangle\Big|_{1L,p=0} = \frac{3\lambda^2}{32\pi^2}\log\frac{\Lambda^2}{m^2}\,. \tag{415}$$

The calculation proceeds along the same lines

$$\langle p_4 p_3|T|p_2 p_1\rangle\Big|_{1L,p=0} = -\frac{3i\lambda^2}{2}\int\frac{\mathrm{d}^4k}{(2\pi)^4}\frac{1}{(k^2-m^2+i\epsilon)^2} = \frac{3\cdot 2\pi^2\lambda^2}{2\cdot 16\pi^4}\int_0^\Lambda \mathrm{d}k_E\frac{k_E^3}{(k_E^2+m^2)^2}\,. \tag{416}$$

This integral can now be calculated e.g. using Mathematica or using a substitution $y = k_E^2$

$$\langle p_4 p_3|T|p_2 p_1\rangle\Big|_{1L,p=0} = \frac{3\lambda^2}{32\pi}\int_0^{\Lambda^2}\mathrm{d}y\frac{y}{(y+m^2)^2}\,. \tag{417}$$

Using partial-fractioning or integration-by-parts we can write

$$\langle p_4 p_3|T|p_2 p_1\rangle\Big|_{1L,p=0} = \frac{3\lambda^2}{32\pi}\Big(\frac{m^2}{m^2+y}+\log(m^2+y)\Big)_0^{\Lambda^2} = \frac{3\lambda^2}{32\pi}\Big(\frac{m^2}{\Lambda^2+m^2}-1+\log\frac{m^2+\Lambda^2}{m^2}\Big) \tag{418}$$

7.1.2 Dimensional regularisation

A big downside of cut-off regularisation is that it breaks Lorentz invariance until we set $\Lambda\to\infty$. Combined with the fact that it makes integrals more complicated, it is no surprise that it is rarely used in practical calculations. Instead, we shift the spacetime dimension away from four, i.e. we work in $d = 4-2\epsilon$ dimensions. This can be formalised but what matters for us is that it regulates the divergences. We write

$$\langle\Omega|T\{\phi(x)\phi(y)\}|\Omega\rangle\Big|_{1L} = \lambda\int\frac{\mathrm{d}^dk}{(2\pi)^d}\frac{i}{k^2-m^2} = -\lambda\int\frac{\mathrm{d}^dk_E}{(2\pi)^d}\frac{1}{k_E^2+m^2} = -\lambda\int\frac{\mathrm{d}\Omega^{(d)}}{(2\pi)^d}\mathrm{d}k\frac{k^{d-1}}{k^2+m^2}\,. \tag{419}$$

The d-dimensional spherical integral can be solved as

$$\int\mathrm{d}\Omega^{(d)} = \frac{2\pi^{d/2}}{\Gamma(d/2)}\,. \tag{420}$$

Proof

Consider the following trick that uses the normalisation of the Gaussian distribution

$$\begin{aligned}(\sqrt{\pi})^d &= \Bigg(\int\mathrm{d}x\ \mathrm{e}^{-x^2}\Bigg)^d = \int\mathrm{d}^dx\ \mathrm{e}^{-\vec{x}^2} = \int\mathrm{d}\Omega^{(d)}\int_0^\infty\mathrm{d}x\ x^{d-1}\mathrm{e}^{-x^2}\\ &= \int\mathrm{d}\Omega^{(d)}\frac{1}{2}\int_0^\infty\mathrm{d}y\ y^{d/2-1}\mathrm{e}^{-y} = \int\mathrm{d}\Omega^{(d)}\frac{1}{2}\Gamma(d/2)\,,\end{aligned} \tag{421}$$

with $y = x^2$.

We now have

$$\langle\Omega|T\{\phi(x)\phi(y)\}|\Omega\rangle\Big|_{1L} = -\frac{\lambda}{(4\pi)^{d/2}}\Gamma\Big(1-\tfrac{d}{2}\Big)(m^2)^{d/2-1}\,. \tag{422}$$

Since this effectively changes the dimension of the coupling or the action, it is customary to add a factor $\mu^{2\epsilon}$

$$\langle\Omega|T\{\phi(x)\phi(y)\}|\Omega\rangle\Big|_{1L} = -m^2\Big(\frac{\mu^2}{m^2}\Big)^\epsilon\frac{\lambda}{(4\pi)^{2-\epsilon}}\Gamma(\epsilon-1) = \frac{m^2\lambda}{16\pi^2}\frac{1}{\epsilon}+\mathcal{O}(\epsilon^0)\,. \tag{423}$$

Once again, this does not solve our problem but it makes it manifest as a pole in $1/\epsilon$.

Exercise: Show that in dimreg

$$\langle p_4p_3|T|p_2p_1\rangle\Big|_{1L,p=0} = \frac{-3\lambda^2}{(4\pi)^{2-\epsilon}}\frac{\Gamma(2-\epsilon)\Gamma(\epsilon-1)}{\Gamma(1-\epsilon)}\Big(\frac{\mu^2}{m^2}\Big)^\epsilon = \frac{3\lambda^2}{16\pi^2}\frac{1}{\epsilon}+\mathcal{O}(\epsilon^0)\,. \tag{424}$$

7.2 Renormalisation

Now that we have the divergences explicit, we can think about fixing them. So far we have just assumed that our semi-classical construction of the fields ϕ was a good one. But in reality, there is no physical interpretation in the parameters of the Lagrangian, be they ϕ, m, or λ. The only thing that is physical are $\mathcal{S}$ matrix elements and the location of the pole of the propagator (which we called mass before). We have now found $\mathcal{S}$ matrix elements that made no sense whatsoever. Is it therefore maybe possible that our choice of parameters in $\mathcal{L}$ were bad?

Since we would have to measure these parameters by studying $\mathcal{S}$ matrix elements, we can not really *predict* $\phi\phi\to\phi\phi$ scattering since we do not yet know λ. Would it therefore be possible to first *measure* $\phi\phi\to\phi\phi$, calculate λ and then measure for example $\phi\phi\to\phi\phi\phi\phi$ as a prediction? The parameter λ in the Lagrangian is meaningless, the only thing that matters are relations between observables; λ is just a convenient intermediary.

Let us therefore add labels to our old Lagrangian to indicate that these quantities were a first guess, called bare quantities

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\phi_0)^2 - \frac{1}{2}m_0^2\phi_0^2 - \frac{\lambda_0}{4!}\phi_0^4\,. \tag{425}$$

When we calculated $\mathcal{S}$ matrix elements with these bare quantities, we would them to depend on the regulator. To cancel this dependency, the bare parameters need to depend on the regulator, denoted by $\mathcal{R}$, themselves, i.e.

$$\mathcal{L}_0(\mathcal{R}) = \frac{1}{2}(\partial_\mu\phi_0(\mathcal{R}))^2 - \frac{1}{2}m_0(\mathcal{R})^2\phi_0(\mathcal{R})^2 - \frac{\lambda_0(\mathcal{R})}{4!}\phi_0(\mathcal{R})^4\,. \tag{426}$$

This means that the bare coupling λ_0, mass m_0, and field ϕ_0 are meaningless, so let us relate them to meaningful quantities

$$\phi_0(\mathcal{R}) = Z_\phi^{1/2}(\mathcal{R})\;\phi\,,\quad m_0(\mathcal{R}) = Z_m(\mathcal{R})\;m\,,\quad \lambda_0(\mathcal{R}) = Z_\lambda(\mathcal{R})\;\lambda\,. \tag{427}$$

We want the renormalised quantities to be physical, i.e. not depend on $\mathcal{R}$ which means that the Z factors also need to depend on the regulator. These quantities are called renomalisation constant and more specifically, Z_ϕ is the field strength renormalisation, Z_m is the mass renormalisation, and Z_λ is the coupling renormalisation.

When expressing the bare Lagrangian using renormalised objects, we find

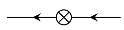
$$\mathcal{L}_0 = \frac{1}{2}Z_\phi(\partial_\mu\phi)^2 - Z_\phi Z_m^2\frac{1}{2}m^2\phi^2 - Z_\phi^2 Z_\lambda\frac{\lambda}{4!}\phi^4\,. \tag{428}$$

Expanding the $Z_i = 1+\lambda\delta Z_i+\mathcal{O}(\lambda^2)$, we can rearrange this to be

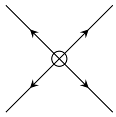
$$\begin{aligned}\mathcal{L}_0 = \frac{1}{2}(\partial_\mu\phi)^2 - \frac{1}{2}m^2\phi^2 - \frac{\lambda}{4!}\phi^4 \\ + \lambda\;\delta Z_\phi\frac{1}{2}(\partial_\mu\phi)^2 - \lambda(\delta Z_\phi+2\delta Z_m)\frac{1}{2}m^2\phi^2 - \lambda(\delta Z_\lambda+2\delta Z_\phi)\frac{\lambda}{4!}\phi^4+\mathcal{O}(\lambda^2)\,.\end{aligned} \tag{429}$$

The first line of this is just the same as before and we have the same Feynman rules to use for our one-loop calculation. The new terms essentially give rise to new Feynman rules that are $\mathcal{O}(\lambda)$. The δZ_i

are usually referred to as counterterm and therefore the resulting vertices are called counterterm vertices. We therefore complement our set of Feynman rules by



$$p \; = i\lambda\Big(p^2\delta Z_\phi - m^2(\delta Z_\phi + 2\delta Z_m)\Big)\,, \tag{430}$$



$$= -i\lambda^2\Big(2\delta Z_\phi + \delta Z_\lambda\Big)\,. \tag{431}$$

When adding these to our calculations, we need to be careful and expand to the same order in λ for each term.

7.2.1 Cut-off regularisation

We have

$$\langle\Omega|T\{\phi(x)\phi(y)\}|\Omega\rangle\Big|_{1L+CT} = -\frac{m^2\lambda}{32\pi^2}\left(\frac{\Lambda^2}{m^2} - \log\frac{\Lambda^2}{m^2}\right) + \lambda\Big(p^2\delta Z_\phi - m^2(\delta Z_\phi + 2\delta Z_m)\Big) + \mathcal{O}\big((\Lambda/m)^0\big)\,, \tag{432}$$

$$\langle p_4 p_3|T|p_2 p_1\rangle\Big|_{1L+CT,p=0} = \frac{3\lambda^2}{32\pi^2}\log\frac{\Lambda^2}{m^2} - \lambda^2\Big(2\delta Z_\phi + \delta Z_\lambda\Big) + \mathcal{O}\big((\Lambda/m)^0\big)\,. \tag{433}$$

We can now all but read of the counterterms. For $\langle\Omega|T\{\phi(x)\phi(y)\}|\Omega\rangle$, we want this to hold regardless of what m^2 and p^2 are, so just collect coefficients. The only solution is

$$\delta Z_\phi = 0\,, \tag{434a}$$

$$\delta Z_m = -\frac{1}{64\pi^2}\left(\frac{\Lambda^2}{m^2} - \log\frac{\Lambda^2}{m^2}\right), \tag{434b}$$

$$\delta Z_\lambda = \frac{3}{32\pi^2}\log\frac{\Lambda^2}{m^2}\,. \tag{434c}$$

Note that this choice was not unique. We could have added more or less finite terms as long as we remove the Λ dependence. The choice we made is called the renormalisation scheme and it is possible to convert between different schemes. For example, we could have chosen Z_λ such that the one-loop four-point function is not just finite but zero for $p = 0$.

7.2.2 Dimensional regularisation

We can do the same here

$$\langle\Omega|T\{\phi(x)\phi(y)\}|\Omega\rangle\Big|_{1L+CT} = \frac{m^2\lambda}{16\pi^2}\frac{1}{\epsilon} + \lambda\Big(p^2\delta Z_\phi - m^2(\delta Z_\phi + 2\delta Z_m)\Big) + \mathcal{O}(\epsilon^0)\,, \tag{435}$$

$$\langle p_4 p_3|T|p_2 p_1\rangle\Big|_{1L+CT,p=0} = \frac{3\lambda^2}{16\pi^2}\frac{1}{\epsilon} - \lambda^2\Big(2\delta Z_\phi + \delta Z_\lambda\Big) + \mathcal{O}(\epsilon^0)\,, \tag{436}$$

and find

$$\delta Z_\phi = 0\,, \tag{437a}$$

$$\delta Z_m = \frac{1}{32\pi^2}\frac{1}{\epsilon}\,, \tag{437b}$$

$$\delta Z_\lambda = \frac{3}{16\pi^2}\frac{1}{\epsilon}\,. \tag{437c}$$

This scheme is famous enough to have its own name, minimal subtraction (MS).

There is one more modification we would like to make. Consider the renormalised result expanded to the finite term

$$\langle p_4 p_3|T|p_2 p_1\rangle\Big|_{1L+CT,p=0} = -\frac{3\lambda^2}{16\pi^2}\Big(\underbrace{\gamma_E - \log(4\pi)}_{} - \log\frac{\mu^2}{m^2}\Big)\,, \tag{438}$$

with Euler's constant (not be confused with e

$$\gamma_E = \frac{\mathrm{d}}{\mathrm{d}x}\Gamma(1-x)\Big|_{x=0} = 0.577216\ldots \tag{439}$$

This and the $\log(4\pi)$ are artefacts of our calculation and not physical. They are therefore almost universally removed by modifying the renormalisation constants to be

$$\delta Z_\phi = 0\,, \tag{440a}$$

$$\delta Z_m = \frac{1}{32\pi^2}\frac{1}{\epsilon}(4\pi)^\epsilon \mathrm{e}^{-\gamma_E\epsilon}\,, \tag{440b}$$

$$\delta Z_\lambda = \frac{3}{16\pi^2}\frac{1}{\epsilon}(4\pi)^\epsilon \mathrm{e}^{-\gamma_E\epsilon}\,. \tag{440c}$$

This scheme is now called modified minimal subtraction ($\overline{\mathrm{MS}}$).

7.3 Calculation of $\phi\phi \to \phi\phi$ at non-zero momentum

Full calculation of the process
We have used the $p = 0$ case to fix the coupling but we can still calculate the one-loop corrections to $\phi\phi \to \phi\phi$ scattering. To do this, we write down the *full* diagram, including momentum dependence

$$\langle p_4 p_3 | T | p_2 p_1 \rangle\Big|_{1L} = \tag{441}$$

$$= \frac{-i\lambda^2}{2}\underbrace{\mu^{2\epsilon}\int \frac{\mathrm{d}^d k}{(2\pi)^d}\frac{1}{k^2-m^2}\frac{1}{(k+p_1+p_2)^2-m^2}}_{\mathcal{I}} + (p_2 \leftrightarrow p_3) + (p_2 \leftrightarrow p_4)\,. \tag{442}$$

To solve this loop integral we employ a trick called Feynman parametrisation

$$\frac{1}{AB} = \int_0^\infty \mathrm{d}x\mathrm{d}y \frac{\delta(\cdots)}{(xA+yB)^2}\,. \tag{443}$$

The delta function can either be $\delta(1-x-y)$ or $\delta(1-x)$. We will choose the former. We can now write with $Q = p_1 + p_2$

$$\mathcal{I} = \mu^{2\epsilon}\int \mathrm{d}x\mathrm{d}y \frac{\mathrm{d}^d k}{(2\pi)^d}\frac{\delta(1-x-y)}{\Big(x\big[k^2-m^2\big] + y\big[(k+Q)^2-m^2\big]\Big)^2} \tag{444}$$

$$= \mu^{2\epsilon}\int_0^1 \mathrm{d}y \int \frac{\mathrm{d}^d k}{(2\pi)^d}\frac{1}{\Big((k+Qy)^2 - m^2 + (1-y)yQ^2\Big)^2}\,. \tag{445}$$

In the last step, we have completed the square in the denominator and can now shift $k \to k - Qy$ to have once again

$$\mathcal{I} = \mu^{2\epsilon}\int_0^1 \mathrm{d}y \int \frac{\mathrm{d}^d k}{(2\pi)^d}\frac{1}{\Big(k^2 - m^2 + (1-y)yQ^2\Big)^2} = \mu^{2\epsilon}\frac{\Gamma(\epsilon)}{(2\pi)^d}\int_0^1 \mathrm{d}y\,\Big(m^2 + Q^2(y-1)y\Big)^{-\epsilon}\,. \tag{446}$$

This integral can be evaluated for example using Mathematica

$$\mathcal{I} = \Big(\frac{\mu^2}{m^2}\Big)^\epsilon \frac{\Gamma(-1+\epsilon)}{4(4\pi)^{d/2}}\frac{\beta^2-1}{\beta}\left({}_2F_1\left[\begin{matrix}1, 2-2\epsilon\\ 2-\epsilon\end{matrix}; \frac{\beta-1}{2\beta}\right] - {}_2F_1\left[\begin{matrix}1, 2-2\epsilon\\ 2-\epsilon\end{matrix}; \frac{\beta+1}{2\beta}\right]\right) \tag{447}$$

$$= \frac{1}{16\pi^2}\left(\frac{1}{\epsilon} - \gamma_E + \log(4\pi) + 2 + \log\frac{\mu^2}{m^2} + \beta\log\frac{\beta-1}{\beta+1} + \mathcal{O}(\epsilon)\right)\,, \tag{448}$$

where we have introduced the Gauss-hypergeometric function and $\beta = \sqrt{1-4m^2/Q^2}$. Adding all diagrams, we have with β_s, β_t, and β_u defined by their Mandelstam variables

$$\langle p_4 p_3|T|p_2 p_1\rangle\Big|_{1L} = \frac{\lambda^2}{16\pi^2}\Bigg(\frac{3}{\epsilon} - 3\gamma_E + 3\log(4\pi) + 6 + 3\log\frac{\mu^2}{m^2} + \beta_s \log\frac{\beta_s - 1}{\beta_s + 1} + \beta_t \log\frac{\beta_t - 1}{\beta_t + 1} + \beta_u \log\frac{\beta_u - 1}{\beta_u + 1} + \mathcal{O}(\epsilon)\Bigg). \tag{449}$$

Renormalising the coupling in the $\overline{\text{MS}}$ scheme, we arrive at

$$\langle p_4 p_3|T|p_2 p_1\rangle\Big|_{1L} = \frac{\lambda^2}{16\pi^2}\Bigg(6 + 3\log\frac{\mu^2}{m^2} + \beta_s \log\frac{\beta_s - 1}{\beta_s + 1} + \beta_t \log\frac{\beta_t - 1}{\beta_t + 1} + \beta_u \log\frac{\beta_u - 1}{\beta_u + 1}\Bigg). \tag{450}$$

7.4 Renormalisibility

You may now wonder whether renormalisation is always possible. Can we always find finitely many Z_i to fix all divergences of our theory to any order in perturbation theory? If so, the theory is predictive once all n parameters of the Lagrangian have been fixed using n measurements. Any such theory is called renormalisable and one can show that the Standard Model of particle physics (as well as QED and QCD separately) is renormalisable. However, certain theories like the Fermi description of the beta decay or the simplest quantum theory of gravity are not renormalisable.

7.4.1 For scalar theories

The first step of showing whether a theory is renormalisable is to consider the superficial degree of divergence of the diagrams it can generate. The singularities we need to remove are due to the large k behaviour and we have seen above that each loop gives a factor of d^4k and each propagator a factor of $1/(k^2-m^2) \sim k^{-2}$ Consider therefore a diagram at L loop with P internal lines. The degree of divergence D is defined as the scaling of the integrand for large k and for our scalar theory it is

$$D = 4L - 2P\,. \tag{451}$$

If $D < 0$, the resulting integral is superficially finite and we can ignore it for our discussion of renormalisation. If $D > 0$, the integral is definitely divergent and needs to be considered. The case of $D = 0$ cannot be decided through power-counting and the integral actually needs to be computed (hence the *superficial*).

One can show that for any Feynman diagram with V vertices (Euler's formula)

$$L = P - V + 1 \tag{452}$$

You may have seen this written as the Euler characteristic $\chi = V - E + F = 2$ for a polyhedron with V vertices, E edges and $F = L + 1$ faces (those included by the loop and the outside).

Proof of Euler's formula for graphs
Begin with a single vertex, i.e. $V = 1$, $P = 0$, $L = 0$. This satisfies Euler's formula trivially. Using induction, we can now either

- add a vertex by connecting it with a propagator/edge to the existing ones ($V \to V+1$, $P \to P+1$). (452) remains satisfied.
- connect two existing vertices with a propagator/edge. This creates a new loop/face ($P \to P+1$, $L \to L+1$). (452) remains satisfied.

If we work in a ϕ^n theory, any vertex needs to be connected to n different lines. These could either be one of the P internal (in which case they are shared between two vertices) or N external lines Mathematically,

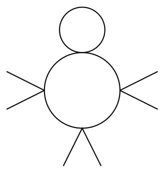
$$2P + N = nV \tag{453}$$

Therefore, we find for D

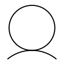
$$D = (n-4)V + 4 - N\,. \tag{454}$$

For ϕ^3 and ϕ^4, we can easily see that the only divergent diagrams are $N \le 4$. Since there are only finitely many such diagrams we can always subtract the divergence with a counterterm.

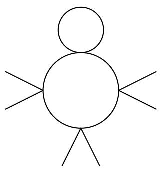
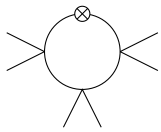
Note that we have to go through the above discussion step-by-step, order-by-order. We can only consider $L = 2$ once we have calculated all counterterms for $L = 1$ etc. This is because of diagrams like

$\mathcal{M}^{(2)}(\phi\phi\phi \to \phi\phi\phi) \supset$  . (455)

Superficially, this diagram ($n = 4$, $L = 2$, $N = 6$, $V = 4$, $P = 5$) should be finite with $D = -2$. However, if we only consider the tadpole that is attached to the 'main' loop

$\mathcal{M}^{(2)}(\phi\phi\phi \to \phi\phi\phi) \supset$  . (456)

This sub-diagram ($L = 1$, $N = 2$, $V = 1$, $P = 1$), which we could have decided to calculate first, has $D = 2$ and is clearly divergent. However, it also renormalised by the counterterm for Z_m and Z_ϕ so that

$\mathcal{M}^{(2)}(\phi\phi\phi \to \phi\phi\phi) \supset$  $+$  $=$ finite. (457)

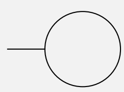
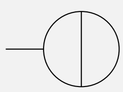
This concept of sub-diagrams being renormalised order-by-order is crucial to the concept of renormalisibility. We can therefore confidently state that for $n = 4$ the theory is renormalisable because $N = 1$ or $N = 3$ are excluded from symmetry.

Exercise: Try drawing a more complicated diagram with $N_e > 4$ external lines and argue why the diagram is finite.

For the $n = 3$ case we already know that we only need to consider $N \le 4$ and we can introduce counterterms Z_ϕ and Z_m to handle the $N = 2$ case (which can be divergent for $V = 2$ since $D = 2 - V$). Crucially, at higher loops the number of vertices grows and therefore D further decreases. This means that only finitely many diagrams are divergent.

The $N = 1$ case

For the $N = 1$ case, we get these following divergent diagrams

$\Delta =$  $+$  . (458)

Calculating the first term using cut-off regularisation, we find

$$\Delta = \frac{\lambda}{2}\int \frac{\mathrm{d}^4 k}{(2\pi)^4}\frac{1}{k^2 - m^2 + i\epsilon} + \mathcal{O}(\lambda^2) = \frac{\lambda}{2}\frac{i}{16\pi^2}\Lambda^2\,. \tag{459}$$

n	divergent N				$[\lambda]$
	$L=1$	$L=2$	$L=3$	$L=4$	
3	1, 2	1	none	none	GeV^1
4	2, 4	2, 4	2, 4	2, 4	GeV^0
5	≤ 6	≤ 7	≤ 8	≤ 9	GeV^{-1}
6	≤ 8	≤ 10	≤ 12	≤ 14	GeV^{-2}

Figure 8: A list of ϕ^n theories and the number of divergent diagrams required at each loop order.

There is no operator in our Lagrangian to fix this, meaning we have to introduce a new term $\mathcal{L} \supset Y\phi$. The resulting Feynman rule is

$$= -iY \,. \tag{460}$$

The renormalisation of this operator Z_Y fixes the divergent diagrams but the operator also leads to a non-zero vev

$$\langle 0|\phi(0)|0\rangle = iY + \Delta \,. \tag{461}$$

As long as the loop diagrams are also imaginary (which they are), the vev can be forced back to be zero, i.e. $\langle 0|\phi(0)|0\rangle = 0$, while keeping Y real.

For the $n = 5$ case we have a problem. Because $D = -V + 4 - N$, the number of counterterms we need, i.e. the number of distinctly divergent N, grows as we increase the number of loops or vertices. This means that, as we go higher in the perturbative expansion, we need ever more counterterms, severely limiting the predictive power of our theory.

Base on the above discussion, we can define three types of theories (cf. also Figure 8).

- super-renormalisable theories like $n = 3$ where only finitely many diagrams are divergent (not counting divergent sub-diagrams).
- renormalisable theories like $n = 4$ where infinitely many diagrams are divergent (not counting divergent sub-diagrams) but the divergence can be remedied order-by-order using finitely many counterterms.
- non-renormalisable theories like $n \leq 5$ where we need infinitely many counterterms.

Non-renormalisable theories were long-held to be useless because of their lack of predictive power. However, if we view these theories merely as describing the low-energy behaviour of some unknown theory we can still use them as an effective field theory (EFT) description.

We can view this description also in terms of the mass-dimension of the operator or coupling. As we have discussed in the very beginning of the course, the action is dimensionless, meaning that $\mathcal{L}$ has mass-dimension $[\mathcal{L}] = \text{GeV}^4$. Since $[\partial] = [m] = \text{GeV}$, $[\phi] = \text{GeV}$ as well. This means that the coupling of the renormalisable ϕ^4 theory is $[\lambda] = 1$ and for the ϕ^3 theory $[\lambda] = \text{GeV}$. For the non-renormalisable ϕ^5 theory we have $[\lambda] = \text{GeV}^{-1}$ etc. Therefore, there is a direct mapping between the renormalisibility of an operator and its mass dimension.

7.4.2 For QED

For QED the arguments work exactly the same way except that the mass dimension due to P changes. For fermions, we have $S \sim 1/k$ and for photons $D_F^{\mu\nu} \sim 1/k^2$. Therefore,

$$D = 4L - P_e - 2P_\gamma = 4 - N_\gamma - \frac{3}{2}N_e \,, \tag{462}$$

for a diagram with N_γ external photons and N_e external electrons. Therefore, we deduce that there are up to seven divergent amplitudes (since N_e needs to be even), as shown in Figure 9.

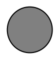
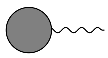
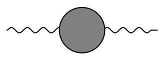
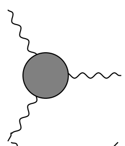
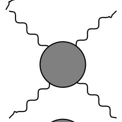
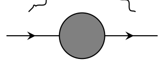
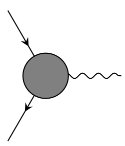
n_e	N_γ	$\mathcal{M}$	D	
0	0		4	unobservable vacuum shift
0	1		3	vanishes because of $\langle j^\mu \rangle$
0	2		2	Z_A
0	3		1	zero by Furry theorem
0	4		0	actually finite
2	0		1	Z_ψ
2	1		0	Z_e

Figure 9: Superficiality divergent diagrams of QED and their fate

A The Lehmann-Symanzik-Zimmermann reduction formula

For the full field $\phi(x)$, the EoM is not just $(\partial^2 + m^2)\phi(x)$. However, we can still calculate this as

$$\int \mathrm{d}^3x\ \mathrm{e}^{ik\cdot x}(\partial^2 + m^2)\phi(x) = \int \mathrm{d}^3x\ \mathrm{e}^{ik\cdot x}(\partial_t^2 - \vec{\nabla}^2 + m^2)\phi(x) \overset{*}{=} \int \mathrm{d}^3x\ (\vec{\nabla}^2\mathrm{e}^{ik\cdot x})(\partial_t^2 + m^2)\phi(x) \tag{463}$$

$$= \int \mathrm{d}^3x\ \mathrm{e}^{ik\cdot x}(\partial_t^2 + \vec{k}^2 + m^2)\phi(x) = \int \mathrm{d}^3x\ \mathrm{e}^{ik\cdot x}(\partial_t^2 + E_{\vec{k}}^2)\phi(x)\,, \tag{464}$$

where we have used integration-by-parts at $*$ for the $\vec{\nabla}$ term which therefore switches signs. This makes no assumption on the structure of the field except that it falls of quickly enough so that the boundary conditions do not contribute. Further, we have used that $\vec{k}^2 + m^2 = E_{\vec{k}}^2$. Consider now

$$\begin{aligned} i\mathrm{e}^{ik\cdot x}(\partial_t^2 + E_{\vec{k}}^2) &= \mathrm{e}^{ik\cdot x}(E_{\vec{k}}\partial_t + i\partial_t^2 + iE_{\vec{k}}^2 - E_{\vec{k}}\partial_t) \\ &= \mathrm{e}^{ik\cdot x}(E_{\vec{k}} + i\partial_t)\partial_t + \mathrm{e}^{ik\cdot x}iE_{\vec{k}}(E_{\vec{k}} + i\partial_t) \\ &= \mathrm{e}^{ik\cdot x}(E_{\vec{k}} + i\partial_t)\partial_t + (\partial_t\mathrm{e}^{ik\cdot x})(E_{\vec{k}} + i\partial_t) = \partial_t\mathrm{e}^{ik\cdot x}(E_{\vec{k}} + i\partial_t)\,, \end{aligned} \tag{465}$$

where the derivative always acts to its right. This means our original expression becomes

$$\int \mathrm{d}^3x\ \mathrm{e}^{ik\cdot x}(\partial^2 + m^2)\phi(x) = -i\int \mathrm{d}^3x\ \partial_t\Big[\mathrm{e}^{ik\cdot x}(E_{\vec{k}} + i\partial_t)\phi(x)\Big]\,. \tag{466}$$

The field $\phi(x)$ here is still the full interacting field which we know little about. However, if we integrate $t = -\infty$ to $t = +\infty$, the derivative turns the expression into its boundary terms

$$\int_{-\infty}^{+\infty} \mathrm{d}t \int \mathrm{d}^3x\ \mathrm{e}^{ik\cdot x}(\partial^2 + m^2)\phi(x) = -i\int \mathrm{d}^3x\ \mathrm{e}^{ik\cdot x}(E_{\vec{k}} + i\partial_t)\phi(x)\Big|_{t=-\infty}^{t=+\infty}\,. \tag{467}$$

In these limits, we actually do understand the field ϕ as the in and out fields that fulfil the free KG equation and can be written in terms of a and $a^\dagger$ operators. However, we need to keep renormalisation in mind. The free field[9] ϕ_f is related to the interacting field through (427), modifying (134) and (137).

$$Z_\phi^{1/2}\phi_{\mathrm{in}} = \lim_{t\to-\infty}\phi(t) = \int \frac{\mathrm{d}^3k}{(2\pi)^3}\frac{1}{\sqrt{2E_{\vec{k}}}}\Big[a_{\mathrm{in}}(\vec{k})\mathrm{e}^{-ik\cdot x} + a_{\mathrm{in}}^\dagger(\vec{k})\mathrm{e}^{ik\cdot x}\Big]\,, \tag{468}$$

$$Z_\phi^{1/2}\phi_{\mathrm{out}} = \lim_{t\to+\infty}\phi(t) = \int \frac{\mathrm{d}^3k}{(2\pi)^3}\frac{1}{\sqrt{2E_{\vec{k}}}}\Big[a_{\mathrm{out}}(\vec{k})\mathrm{e}^{-ik\cdot x} + a_{\mathrm{out}}^\dagger(\vec{k})\mathrm{e}^{ik\cdot x}\Big]\,. \tag{469}$$

Let us therefore calculate for a free field ϕ_f which we will either identify with ϕ_{in} or ϕ_{out}

$$\begin{aligned} \mathrm{e}^{ik\cdot x}(E_{\vec{k}} + i\partial_t)\phi_f(x) &= \int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\vec{p}}}}\mathrm{e}^{ik\cdot x}(E_{\vec{k}} + i\partial_t)\Big[a(\vec{p})\mathrm{e}^{-ip\cdot x} + a^\dagger(\vec{p})\mathrm{e}^{ip\cdot x}\Big] \\ &= \int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\vec{p}}}}\Big[a(\vec{p})(E_{\vec{k}} + E_{\vec{p}})\mathrm{e}^{-i(p-k)\cdot x} + a^\dagger(\vec{p})(E_{\vec{k}} - E_{\vec{p}})\mathrm{e}^{-i(-p-k)\cdot x}\Big]\,. \end{aligned} \tag{470}$$

Integrating over d^3x and using (88)

$$\begin{aligned} \int \mathrm{d}^3x\ \mathrm{e}^{ik\cdot x}(E_{\vec{k}} + i\partial_t)\phi_f(x) &= \int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\vec{p}}}}\Big[a(\vec{p})(E_{\vec{k}} + E_{\vec{p}})(2\pi)^3\mathrm{e}^{-i(E_{\vec{p}}-E_{\vec{k}})t}\delta^{(3)}(\vec{p} + \vec{k}) \\ &\qquad + a^\dagger(\vec{p})(E_{\vec{k}} - E_{\vec{p}})(2\pi)^3\mathrm{e}^{-i(-E_{\vec{p}}-E_{\vec{k}})t}\delta^{(3)}(-\vec{p} + \vec{k})\Big] \\ &= \sqrt{2E_{\vec{k}}}\ a(\vec{k})\,. \end{aligned} \tag{471}$$

This means we have just found another way of expressing the destruction operator a. We can substitute this into our expression and use a_{in} for $t \to -\infty$ and a_{out} for $t \to +\infty$ (using d^4x instead of $\mathrm{d}t\mathrm{d}^3x$)

$$\begin{aligned} \int \mathrm{d}^4x\ \mathrm{e}^{ik\cdot x}(\partial^2 + m^2)\phi(x) &= -iZ_\phi^{-1/2}\sqrt{2E_{\vec{k}}}\Big(a_{\mathrm{out}}(\vec{k}) - a_{\mathrm{in}}(\vec{k})\Big)\,, \\ \int \mathrm{d}^4x\ \mathrm{e}^{-ik\cdot x}(\partial^2 + m^2)\phi(x) &= +iZ_\phi^{-1/2}\sqrt{2E_{\vec{k}}}\Big(a_{\mathrm{out}}^\dagger(\vec{k}) - a_{\mathrm{in}}^\dagger(\vec{k})\Big)\,. \end{aligned} \tag{472}$$

[9] Here we will use f as the subscript rather than 0 to avoid confusion with the bare field

In the limit of the free field, the two operators are identical so that this vanishes – as expected by the EoM for $\phi_f(x)$.

In (142), we have seen how to calculate the probability of a scattering from the amplitude

$$P \sim |{}_o\langle f|i\rangle_i|^2 \,. \tag{473}$$

The in-state $|i\rangle_i$ (out-state $|f\rangle_o$) is created using $a^\dagger_{\text{in}}$ ($a^\dagger_{\text{out}}$) from the vacuum $|\Omega\rangle$. For a process of m particles with momenta q_k to n particles with momenta p_j, we have

$$|i\rangle_i = \sqrt{2E_{\vec{q}_1}}a^\dagger_{\text{in}}(q_1)\ \cdots\ \sqrt{2E_{\vec{q}_m}}a^\dagger_{\text{in}}(q_m)|\Omega\rangle = \left(\prod_{k=1}^{m}\sqrt{2E_{\vec{q}_k}}a^\dagger_{\text{in}}(q_k)\right)|\Omega\rangle \,, \tag{474}$$

$$\langle f|_o = \langle\Omega|\sqrt{2E_{\vec{p}_1}}a_{\text{out}}(p_1)\ \cdots\ \sqrt{2E_{\vec{p}_n}}a_{\text{out}}(p_n) = \langle\Omega|\left(\prod_{j=1}^{n}\sqrt{2E_{\vec{p}_j}}a_{\text{out}}(p_j)\right) . \tag{475}$$

The amplitude therefore becomes

$${}_o\langle f|i\rangle_i = \langle\Omega|\left(\prod_{j=1}^{n}\sqrt{2E_{\vec{p}_j}}a_{\text{out}}(p_j)\right)\left(\prod_{k=1}^{m}\sqrt{2E_{\vec{q}_k}}a^\dagger_{\text{in}}(q_k)\right)|\Omega\rangle \,, \tag{476}$$

The operator product is naturally time-ordered so let us enforce this henceforth. We can now replace the $a^\dagger_{\text{in}}$ and a_{out} using (472)

$$\begin{aligned}{}_o\langle f|i\rangle_i = \langle\Omega|T\Bigg\{\prod_{j=1}^{n}\left[iZ_\phi^{-1/2}\int \mathrm{d}^4x_j\ \mathrm{e}^{ip_j\cdot x_j}(\partial_j^2+m^2)\phi(x_j) + \sqrt{2E_{\vec{p}_j}}a_{\text{in}}(\vec{p}_j)\right] \\ \times\prod_{k=1}^{m}\left[iZ_\phi^{-1/2}\int \mathrm{d}^4y_k\ \mathrm{e}^{-iq_k\cdot y_k}(\partial_k^2+m^2)\phi(y_k) + \sqrt{2E_{\vec{q}_k}}a^\dagger_{\text{out}}(\vec{q}_k)\right]\Bigg\}|\Omega\rangle \,.\end{aligned} \tag{477}$$

Note how the $a^\dagger_{\text{out}}(\vec{q}_k)$ is currently all the way to the right of the expression, even though it is taken at very early time. This means that time-ordering pushes it all the way to the left. Similarly, the $a_{\text{in}}(\vec{p}_j)$ will pushed to the right where it acts on the vacuum $|\Omega\rangle$. Dropping these disconnected terms, we have and identifying the left-hand side with the $\mathcal{S}$ matrix element $\langle f|\mathcal{S}|i\rangle$

$$\begin{aligned}{}_o\langle f|i\rangle_i = \langle f|\mathcal{S}|i\rangle = \int\left[\prod_{j=1}^{n}\mathrm{d}^4x_j\ \frac{i}{\sqrt{Z_\phi}}\mathrm{e}^{ip_j\cdot x_j}(\partial_j^2+m^2)\right]\left[\prod_{k=1}^{m}\mathrm{d}^4y_k\ \frac{i}{\sqrt{Z_\phi}}\mathrm{e}^{-iq_k\cdot y_k}(\partial_k^2+m^2)\right] \\ \times\langle\Omega|T\Big\{\phi(x_1)\cdots\phi(x_n)\cdot\phi(y_1)\cdots\phi(y_m)\Big\}|\Omega\rangle \,.\end{aligned} \tag{478}$$

This result is known as the Lehmann-Symanzik-Zimmermann (LSZ) reduction formula and it is the last missing piece of our discussion. We can transform this into momentum space as well where we replace $\partial_j^2 \to -p_j^2$

$$\begin{aligned}\langle f|\mathcal{S}|i\rangle = \int\left[\prod_{j=1}^{n}\mathrm{d}^4x_j\ \frac{-i}{\sqrt{Z_\phi}}\mathrm{e}^{ip_j\cdot x_j}(p_j^2-m^2)\right]\left[\prod_{k=1}^{m}\mathrm{d}^4y_k\ \frac{-i}{\sqrt{Z_\phi}}\mathrm{e}^{-iq_k\cdot y_k}(q_k^2-m^2)\right] \\ \times\langle\Omega|T\Big\{\phi(x_1)\cdots\phi(x_n)\cdot\phi(y_1)\cdots\phi(y_m)\Big\}|\Omega\rangle \,.\end{aligned} \tag{479}$$

We can re-interpret this by moving the factors of $p_j^2-m^2$ and $\sqrt{Z_\phi}$ to the other side

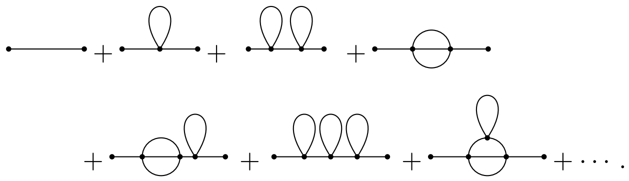
$$\begin{aligned}\left[\prod_{j=1}^{n}\frac{i\sqrt{Z_\phi}}{p_j^2-m^2}\right]\left[\prod_{k=1}^{m}\frac{i\sqrt{Z_\phi}}{q_k^2-m^2}\right]\langle f|\mathcal{S}|i\rangle = \int\prod_{j=1}^{n}\mathrm{d}^4x_j\ \mathrm{e}^{ip_j\cdot x_j}\int\prod_{k=1}^{m}\mathrm{d}^4y_k\ \mathrm{e}^{-iq_k\cdot y_k} \\ \times\langle\Omega|T\Big\{\phi(x_1)\cdots\phi(x_n)\cdot\phi(y_1)\cdots\phi(y_m)\Big\}|\Omega\rangle \,.\end{aligned} \tag{480}$$

This is the relation we have been implicitly using in Section 3 when we related the $\mathcal{S}$ matrix element to the (Fourier-transformed) correlation function.

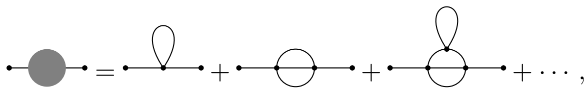
The correlation function $\langle\Omega|T\{\phi\cdots\phi\}|\Omega\rangle$ still contains the non-amputated pieces that we were trying to figure out in (225). In general, these terms exist and do contribute to the off-shell correlation function

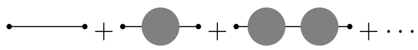
$$\langle\Omega|T\{\phi\cdots\phi\}|\Omega\rangle = \Big(\text{amputated}\Big) \times \Big(\text{non-amputated}\Big)\,. \tag{481}$$

The non-amputated pieces are just the bare propagator $\langle\Omega|T\{\phi_0\phi_0\}|\Omega\rangle$ which we calculated in (406)

$\langle\Omega|T\{\phi_0\phi_0\}|\Omega\rangle =$  (482)

Some of these terms such as the third, fifth, and sixth are different from the others in that they can be cut in half and just expressed through two or more copies. The terms for which this is not possible are called 1PI. If we bundle all the 1PI corrections into a blob, we can write

$-i1\text{PI} =$  (483)

$\langle\Omega|T\{\phi_0\phi_0\}|\Omega\rangle =$ 

$$= \frac{i}{p^2-m_0^2} + \frac{i}{p^2-m_0^2}(-i1\text{PI})\frac{i}{p^2-m_0^2} + \frac{i}{p^2-m_0^2}(-i1\text{PI})\frac{i}{p^2-m_0^2}(-i1\text{PI})\frac{i}{p^2-m_0^2}\,. \tag{484}$$

The geometric series can be summed to result in the propagator

$$\langle\Omega|T\{\phi_0\phi_0\}|\Omega\rangle = \frac{i}{p^2-m_0^2-1\text{PI}}\,. \tag{485}$$

This is to be compared to the equivalent renormalised expression $\langle\Omega|T\{\phi\phi\}|\Omega\rangle = iZ_\phi/(p^2-m^2)$

$$\langle\Omega|T\{\phi_0\phi_0\}|\Omega\rangle = \frac{i}{p^2-m_0^2-1\text{PI}} \sim \frac{iZ_\phi}{p^2-m^2} + \text{regular}\,. \tag{486}$$

Since the LSZ formula requires us to pick out only the singular terms of the correlation function when calculating $\mathcal{S}$ matrix elements, we have

$$\big(\sqrt{Z_\phi}\big)^{n+m}\langle f|\mathcal{S}|i\rangle = \langle\Omega|T\{\phi\cdots\phi\}|\Omega\rangle\Big|_{\text{singular}} = Z_\phi^{n+m}\ \langle\Omega|T\{\phi\cdots\phi\}|\Omega\rangle\Big|_{\text{amputated}}\,. \tag{487}$$

This is exactly what we stated in (226) without proving it

$$\langle f|\mathcal{S}|i\rangle = \big(\sqrt{Z_\phi}\big)^{n+m}\langle\Omega|T\{\phi\cdots\phi\}|\Omega\rangle\Big|_{\text{amputated}}\,. \tag{488}$$

Confusingly this result is also sometimes referred to as the LSZ formula and it is our main recipe for calculating $\mathcal{S}$ matrix elements: calculate the connected and amputated Feynman diagrams using the correlation function, take the external legs on-shell and multiply with $\sqrt{Z_\phi}$ for each particle.

List of acronyms

CCR canonical commutation relation

EoM equations of motion

LHC Large Hadron Collider

KG Klein-Gordon

QM quantum mechanics

QCD quantum chromodynamics

QED quantum electrodynamics

QFT quantum field theory

vev vacuum expectation value

dimreg dimensional regularisation

MS minimal subtraction

$\overline{\text{MS}}$ modified minimal subtraction

LSZ Lehmann-Symanzik-Zimmermann

1PI one-particle irreducible

EFT effective field theory

Index of terms

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