MATH425
Quantum Field Theory
Homework Sheet 3

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Homework 1: Particle annihilation (20 Pts.)

Consider the process $e^+e^- \to \gamma\gamma$ in QED.

a) (3 Pts.) Draw the diagrams contributing to this process and evaluate them using the QED Feynman rules.

b) (10 Pts.) Calculate $|\mathcal{M}|^2$ with $m = 0$.

c) (5 Pts.) Calculate $\mathrm{d}\sigma/\mathrm{d}(\cos\theta)$.

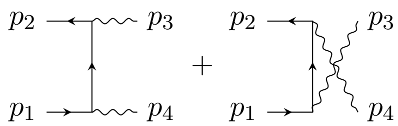
d) (2 Pts.) Is the total cross section defined? If so, what is it? If not, why not?

You may use computer algebra to calculate integrals or simplify expressions. When you do, please make sure to clearly mark it as such.

SOLUTION:

a) There are two diagrams

$\mathcal{M} =$



$$= \bar v(p_2)(-ie\gamma^\mu)\frac{i}{\gamma\cdot(p_1-p_4)-m}(-ie\gamma^\nu)u(p_1)\ \epsilon^*_\mu(p_3)\epsilon^*_\nu(p_4)$$
$$+\bar v(p_2)(-ie\gamma^\nu)\frac{i}{\gamma\cdot(p_1-p_3)-m}(-ie\gamma^\mu)u(p_1)\ \epsilon^*_\mu(p_3)\epsilon^*_\nu(p_4)\,.$$

b) We write $\mathcal{M} = \mathcal{M}_1 + \mathcal{M}_2$

$$\sum|\mathcal{M}_1|^2 = e^4\ \bar v(p_2)\gamma^\mu\frac{1}{\gamma\cdot(p_1-p_4)}\gamma^\nu\underbrace{u(p_1)\bar u(p_1)}_{\gamma\cdot p_1}\gamma^\rho\frac{1}{\gamma\cdot(p_1-p_4)}\gamma^\sigma v(p_2)$$
$$\underbrace{\epsilon_\sigma(p_3)\epsilon^*_\mu(p_3)}_{-\eta_{\mu\sigma}}\underbrace{\epsilon_\rho(p_4)\epsilon^*_\nu(p_4)}_{-\eta_{\nu\rho}}$$
$$= e^4\mathrm{tr}\left((\gamma\cdot p_2)\gamma^\mu\frac{1}{\gamma\cdot(p_1-p_4)}\gamma^\nu(\gamma\cdot p_1)\gamma_\nu\frac{1}{\gamma\cdot(p_1-p_4)}\gamma_\mu\right)\,.$$

The same result for $|\mathcal{M}_2|^2$

$$\sum|\mathcal{M}_2|^2 = e^4\mathrm{tr}\left((\gamma\cdot p_2)\gamma^\nu\frac{1}{\gamma\cdot(p_1-p_3)}\gamma^\mu(\gamma\cdot p_1)\gamma_\mu\frac{1}{\gamma\cdot(p_1-p_3)}\gamma_\nu\right)\,.$$

We can contract $\gamma^\mu(\gamma\cdot p_1)\gamma_\mu$ but also use the cyclic nature of the trace and move the last γ matrix to the start

$$\sum|\mathcal{M}_1|^2 = e^4\mathrm{tr}\left(\underbrace{\gamma_\mu(\gamma\cdot p_2)\gamma^\mu}_{-2\gamma\cdot p_2}\frac{1}{\gamma\cdot(p_1-p_4)}\underbrace{\gamma^\nu(\gamma\cdot p_1)\gamma_\nu}_{-2\gamma\cdot p_1}\frac{1}{\gamma\cdot(p_1-p_4)}\right)$$

$$\sum|\mathcal{M}_2|^2 = e^4\mathrm{tr}\left(\underbrace{\gamma_\nu(\gamma\cdot p_2)\gamma^\nu}_{-2\gamma\cdot p_2}\frac{1}{\gamma\cdot(p_1-p_3)}\underbrace{\gamma^\mu(\gamma\cdot p_1)\gamma_\mu}_{-2\gamma\cdot p_1}\frac{1}{\gamma\cdot(p_1-p_3)}\right)$$

$$\sum|\mathcal{M}_1|^2 = \frac{4e^4}{(p_1-p_4)^4}\mathrm{tr}\Bigg((\gamma\cdot p_2)(\gamma\cdot(p_1-p_4))(\gamma\cdot p_1)(\gamma\cdot(p_1-p_4))\Bigg)$$

$$\sum|\mathcal{M}_2|^2 = \frac{4e^4}{(p_1-p_3)^4}\mathrm{tr}\Bigg((\gamma\cdot p_2)(\gamma\cdot(p_1-p_3))(\gamma\cdot p_1)(\gamma\cdot(p_1-p_3))\Bigg)$$

The trace can be calculated as

$$\begin{aligned}\sum|\mathcal{M}_1|^2 &= \frac{4e^4}{(p_1-p_4)^4}\Bigg(8p_1\cdot(p_1-p_4)p_2\cdot(p_1-p_4) - 4p_1\cdot p_2(p_1-p_4)\cdot(p_1-p_4)\Bigg)\\ &= \frac{8e^4 t}{u}\\ \sum|\mathcal{M}_2|^2 &= \frac{8e^4 u}{t}\,.\end{aligned}$$

Note that it is possible to deduce $|\mathcal{M}_2|^2$ from $|\mathcal{M}_1|^2$ via crossing. We also have to calculate the interference term

$$\begin{aligned}\mathcal{M}_1^*\mathcal{M}_2 &= e^4\bar{u}(p_1)\gamma^\rho\frac{1}{\gamma\cdot(p_1-p_4)}\gamma^\sigma\underbrace{v(p_2)\bar{v}(p_2)}_{\gamma\cdot p_2}\gamma^\nu\frac{1}{\gamma\cdot(p_1-p_3)}\gamma^\mu u(p_1)\underbrace{\epsilon_\sigma(p_3)\epsilon^*_\mu(p_3)}_{-\eta_{\mu\sigma}}\underbrace{\epsilon_\rho(p_4)\epsilon^*_\nu(p_4)}_{-\eta_{\nu\rho}}\\ &= \frac{e^4}{ut}\mathrm{tr}\Bigg((\gamma\cdot p_1)\gamma_\nu(\gamma\cdot(p_1-p_4))\gamma_\mu(\gamma\cdot p_2)\gamma^\nu(\gamma\cdot(p_1-p_3))\gamma^\mu\Bigg)\,.\end{aligned}$$

We can use $\gamma_\nu(\gamma\cdot p)\gamma_\mu(\gamma\cdot q)\gamma^\nu = -2(\gamma\cdot q)\gamma_\mu(\gamma\cdot p)$ and $\gamma_\mu(\gamma\cdot p)(\gamma\cdot q)\gamma^\mu = 4p\cdot q$

$$\begin{aligned}\mathcal{M}_1^*\mathcal{M}_2 &= -\frac{2e^4}{ut}\mathrm{tr}\Big(\gamma^\mu(\gamma\cdot p_1)(\gamma\cdot p_2)\gamma_\mu(\gamma\cdot(p_1-p_4))(\gamma\cdot(p_1-p_3))\Big)\\ &= -\frac{8e^4 p_1\cdot p_2}{ut}\mathrm{tr}\Big((\gamma\cdot(p_1-p_4))(\gamma\cdot(p_1-p_3))\Big)\\ &= -\frac{32e^4 p_1\cdot p_2}{ut}(p_1-p_4)\cdot(p_1-p_3) = -\frac{8e^4 s(s+t+u)}{tu} = 0\,,\end{aligned}$$

since $s+t+u = m_1^2+m_2^2+m_3^2+m_4^2 = 0$. Therefore,

$$\frac{1}{4}\sum|\mathcal{M}|^2 = 2e^4\left(\frac{u}{t}+\frac{t}{u}\right)\,.$$

c) We can now calculate $\mathrm{d}\sigma/\mathrm{d}\cos\theta$. We write

$$\mathrm{d}\sigma = \frac{1}{(2E_a)(2E_b)|\vec{v}|}\mathrm{d}\Phi_2\frac{1}{4}\sum|\mathcal{M}|^2 = \frac{1}{(2E_a)(2E_b)|\vec{v}|}\frac{1}{16\pi^2}\frac{|\vec{p}_3|}{E_{\mathrm{cm}}}\mathrm{d}\Omega\frac{1}{4}\sum|\mathcal{M}|^2\,.$$

Using $E_a = E_b = |\vec{p}_3| = E_3 = \sqrt{s}/2$, $E_{\text{cm}} = \sqrt{s}$, $|\vec{v}| = 1$,

$$\mathrm{d}\sigma = \frac{1}{(2E_a)(2E_b)|\vec{v}|}\frac{1}{16\pi^2}\frac{|\vec{p}_3|}{E_{\text{cm}}}\mathrm{d}\Omega\frac{1}{4}\sum|\mathcal{M}|^2 = \frac{e^4}{32\pi^2}\frac{\mathrm{d}\Omega}{s}\left(\frac{t}{u}+\frac{u}{t}\right) = \frac{\alpha^2}{s}\frac{1+\cos^2\theta}{1-\cos^2\theta}\mathrm{d}\Omega\,.$$

Therefore,

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\cos\theta} = \frac{2\pi\alpha^2}{s}\frac{1+\cos^2\theta}{1-\cos^2\theta}$$

d) The total cross section would mean we have to integrate $\cos\theta$ from -1 to 1 which is not defined. This is due to the fact that we have calculated with massless electrons. Otherwise, we would have had $t - m^2$ in the denominator.