MATH425
Quantum Field Theory
Homework Sheet 2

UNIVERSITY OF LIVERPOOL

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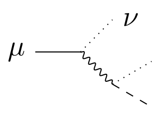
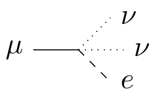
Homework 1: A simple model of the muon decay (10 Pts.)

Context: A muon is a lepton like the electron but its mass is approximately 200× larger ($m_e = 0.511\,\text{MeV}$ vs. $m_\mu = 105\,\text{MeV}$). In the Standard Model of particle physics, the muon can decay into an electron and two neutrinos, i.e. $\mu \to e\nu\bar\nu$. This decay is mediated by a W boson

$\mu \to e\nu\bar\nu =$  . (1)

One can approximate this amplitude by replacing the W with a four-point vertex

$\mu \to e\nu\bar\nu \approx$  (2)

In this exercise sheet we will simplify this further and treat the muon, electron and neutrino as scalar particle ϕ_1, ϕ_2, ϕ_3 respectively.

Consider the following theory

$$\mathcal{L} = \sum_{i=1}^{3}\left[\frac{1}{2}(\partial_\mu\phi_i)(\partial^\mu\phi_i) - \frac{1}{2}m_i^2\phi_i^2\right] - \frac{1}{2!}\lambda\,\phi_1\phi_2\phi_3^2\,, \tag{3}$$

with $m_3 \equiv m_\nu = 0$.

a) (2 Pts.) Write down the Feynman rules of this theory. You do not need to derive these from first principle.

b) (1 Pts.) Calculate the amplitude for the process $\phi_1(P) \to \phi_2(p_1)\phi_3(p_2)\phi_3(p_3)$ as a proxy for $\mu \to e\nu\bar\nu$.

To calculate the total decay rate, we can make use of the fact that if we have a multi-particle phase space like $\mathrm{d}\Phi_3$, we can write this as

$$\mathrm{d}\Phi_3(\phi_1 \to \phi_2\phi_3\phi_3) = \mathrm{d}\Phi_2\Big(\phi_1(P) \to \phi_2(p_1)Q^2\Big)\mathrm{d}\Phi_2\Big(Q^2 \to \phi_3(p_2)\phi_3(p_3)\Big)\frac{1}{2\pi}\mathrm{d}Q^2\,. \tag{4}$$

Here we have introduced a new parameter Q^2 which we can view as a fictitious particle that mediates the process with $Q = p_2 + p_3$. We will now calculate the differential decay rate $\mathrm{d}\Gamma/\mathrm{d}Q^2$.

c) (2 Pts.) To do this, go first into the restframe of Q^2 where $\vec p_2^{\,*} = -\vec p_3^{\,*}$ and calculate

$$\int \mathrm{d}\Phi_2(Q^2 \to \phi_3\phi_3)|\mathcal{M}|^2\,. \tag{5}$$

d) (2 Pts.) Calculate $|\vec{p}_1| = \sqrt{\Lambda}/(2m_1)$ to find Λ in terms of m_1, m_2, and Q^2.

e) (2 Pts.) Calculate $\mathrm{d}\Gamma/\mathrm{d}Q^2$.

f) (1 Pts.) Set $m_2 = 0$ and calculate Γ.

SOLUTION:

a) We have three different propagators and one vertex

$$= \frac{1}{p^2 - m_1^2} \tag{6}$$

$$= \frac{1}{p^2 - m_2^2} \tag{7}$$

$$= \frac{1}{p^2} \tag{8}$$

$$= -i\lambda \tag{9}$$

b) Therefore, the transition amplitude is

$$\mathcal{M}(\phi_1 \to \phi_2\phi_3\phi_3) = -i\lambda\,.$$

c) We begin by considering the two ϕ_3 particles in the frame where $\vec{p}_3 = -\vec{p}_2$. Here, we can consider a fictions particle of mass $\sqrt{Q^2}$ decaying into the two ϕ_3. From the lecture, we have

$$\int \mathrm{d}\Phi(Q^2 \to \phi_3\phi_3) = \int \mathrm{d}\Omega^* \frac{1}{16\pi^2} \frac{|\vec{p}_3^{\,*}|}{\sqrt{Q^2}}\,.$$

We can use the $*$ to indicate that this in this new frame. Here, $|\vec{p}_3| = |\vec{p}_2| = \sqrt{Q^2}/2$ because $m_3 = 0$. Since the matrix element does not depend on $\vec{p}_3$ or $\vec{p}_2$, we can write

$$\int \mathrm{d}\Phi(Q^2 \to \phi_3\phi_3)|\mathcal{M}|^2 = \frac{\lambda^2}{32\pi^2} \int \mathrm{d}\Omega^* = \frac{\lambda^2}{8\pi}\,.$$

Therefore,

$$\mathrm{d}\Gamma = \frac{\lambda^2}{16\pi m_1} \int \mathrm{d}\Phi(\phi_1 \to \phi_2 Q^2)(2\pi)^3 \mathrm{d}Q^2\,.$$

d) Now $|\vec{p}_1|$ is no longer $m_1/2$. Instead, we start with $P = p_1 + (p_2 + p_3)$ with $(p_2 + p_3)^2 = Q^2$ and $\vec{P} = 0$. We can also write

$$Q^2 = (p_2 + p_3)^2 = (P - p_1)^2 = m_1^2 - 2P \cdot p_1 + m_2^2\,. \tag{10}$$

Since $P = (m_1, 0, 0, 0)$, $P \cdot p_1 = m_1 E_1$ and therefore,

$$E_1 = \frac{Q^2 - m_1^2 - m_2^2}{-2m_1}\,. \tag{11}$$

Since $m_2^2 = E_1^2 - |\vec{p}_1|^2$, we can write

$$|\vec{p}_1| = \frac{\sqrt{Q^4 - 2Q^2m_1^2 + m_1^4 - 2Q^2m_2^2 - 2m_1^2m_2^2 + m_2^4}}{2m_1} = \frac{\sqrt{\Lambda}}{2m_1}\,, \tag{12}$$

where we have introduced $\Lambda = Q^4 - 2Q^2m_1^2 + m_1^4 - 2Q^2m_2^2 - 2m_1^2m_2^2 + m_2^4$

e) For the second part, we once again use the equation from the lecture

$$\int \mathrm{d}\Phi(\phi_1 \to \phi_2 M) = \int \mathrm{d}\Omega \frac{1}{16\pi^2} \frac{|\vec{p}_1|}{m_1}\,.$$

Therefore,

$$\frac{\mathrm{d}\Gamma}{\mathrm{d}Q^2} = \frac{\sqrt{\Lambda}}{m_1^3} \frac{\lambda^2}{256\pi^3}\,.$$

f) We now have

$$\frac{\mathrm{d}\Gamma}{\mathrm{d}Q^2} = \frac{m_1 - Q^2}{m_1^3} \frac{\lambda^2}{256\pi^3}\,.$$

Integrating Q^2 from 0 to m_1^2

$$\Gamma = \frac{m_1 \lambda^2}{512\pi^3}$$