MATH425
Quantum Field Theory
Homework Sheet 1

UNIVERSITY OF LIVERPOOL

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Homework 1: Complex scalar field (8 Pts.)

Consider the following Lagrangian of the complex scalar field

$$\mathcal{L} = (\partial_\mu \phi)(\partial^\mu \phi)^* - m^2 \phi \phi^* \,. \tag{1}$$

Compared to the real scalar field we now have two *independent* fields ϕ and ϕ^*.

a) (2 Pts.) Derive the equations of motion for $\phi(x)$ and $\phi^*(x)$.

b) (2 Pts.) Write down the conjugate momenta $\pi(x)$ and $\pi^*(x)$ as well as the Hamiltonian $\mathcal{H}$

c) (2 Pts.) Assume that $\phi(x)$ is a solution to the equations of motion to show that we have a conserved current in

$$j^\mu = -i(\phi \partial^\mu \phi^* - \phi^* \partial^\mu \phi) \,. \tag{2}$$

d) (2 Pts.) Explain why the general complex solution is given by

$$\phi(x) = \int \frac{\mathrm{d}^3 p}{(2\pi)^3} \sqrt{\frac{1}{2E_{\vec p}}} \Big[a_{\vec p}\, \mathrm{e}^{-ip\cdot x} + b^*_{\vec p}\, \mathrm{e}^{ip\cdot x} \Big]_{p^0 = E_{\vec p}} \,, \tag{3}$$

where $a_{\vec p}$ and $b_{\vec p}$ are independent.

SOLUTION:

a) We have two Euler-Lagrange equations

$$0 = \frac{\partial}{\partial x_\mu} \frac{\partial \mathcal{L}}{\partial(\partial^\mu \phi)} - \frac{\partial \mathcal{L}}{\partial \phi} \,,$$
$$0 = \frac{\partial}{\partial x_\mu} \frac{\partial \mathcal{L}}{\partial(\partial^\mu \phi^*)} - \frac{\partial \mathcal{L}}{\partial \phi^*} \,.$$

We have by appropriately raising and lowering the μ index

$$\frac{\partial \mathcal{L}}{\partial(\partial^\mu \phi)} = \partial_\mu \phi^*$$
$$\frac{\partial \mathcal{L}}{\partial(\partial^\mu \phi^*)} = \partial_\mu \phi \,,$$
$$\frac{\partial \mathcal{L}}{\partial \phi} = -m^2 \phi^*$$
$$\frac{\partial \mathcal{L}}{\partial \phi^*} = -m^2 \phi \,.$$

Therefore,

$$0 = \partial^\mu \partial_\mu \phi^* + m^2 \phi^* \,,$$
$$0 = \partial^\mu \partial_\mu \phi + m^2 \phi \,.$$

b) The π and π^* fields are defined as

$$\pi = \frac{\partial \mathcal{L}}{\partial \dot{\phi}} = \dot{\phi}^* \,,$$
$$\pi^* = \frac{\partial \mathcal{L}}{\partial \dot{\phi}^*} = \dot{\phi}$$

And therefore the Hamiltonian with $\phi_1 = \phi$ and $\phi_2 = \phi^*$

$$\mathcal{H} = \sum_i \dot{\phi}_i \pi_i - \mathcal{L} = \dot{\phi}\pi + \dot{\phi}^* \pi^* - \dot{\phi}\dot{\phi}^* + (\vec{\nabla}\phi)(\vec{\nabla}\phi^*) + m^2 \phi\phi^* = \pi^* \pi + (\vec{\nabla}\phi)(\vec{\nabla}\phi^*) + m^2 \phi\phi^* \,,$$

because the fields ϕ and π are assumed to commute.

c) To show this, we calculate $\partial_\mu j^\mu$

$$\begin{aligned}
\partial_\mu j^\mu &= -i\Big(\partial_\mu(\phi \partial^\mu \phi^*) - \partial_\mu(\phi^* \partial^\mu \phi)\Big) \\
&= -i\Big((\partial_\mu \phi)(\partial^\mu \phi^*) + \phi(\partial_\mu \partial^\mu \phi^*) - (\partial_\mu \phi^*)(\partial^\mu \phi) - \phi^*(\partial_\mu \partial^\mu \phi)\Big) \\
&= -i\Big(\phi(\partial_\mu \partial^\mu \phi^*) - \phi^*(\partial_\mu \partial^\mu \phi)\Big) \,.
\end{aligned}$$

Using the equations of motion for ϕ^* and ϕ, we have

$$\partial_\mu j^\mu = +m^2 i\Big(\phi\phi^* - \phi^*\phi\Big) = 0$$

d) This field fulfils the equation of motion

$$\begin{aligned}
\partial^\mu \partial_\mu \phi(x) &= \int \frac{\mathrm{d}^3 p}{(2\pi)^3} \sqrt{\frac{1}{2E_{\vec{p}}}}\; \partial^\mu \partial_\mu \Big[a_{\vec{p}}\, \mathrm{e}^{-ip\cdot x} + b^*_{\vec{p}}\, \mathrm{e}^{ip\cdot x}\Big]_{p^0 = E_{\vec{p}}} \\
&= \int \frac{\mathrm{d}^3 p}{(2\pi)^3} \sqrt{\frac{1}{2E_{\vec{p}}}}\; (-p^2)\Big[a_{\vec{p}}\, \mathrm{e}^{-ip\cdot x} + b^*_{\vec{p}}\, \mathrm{e}^{ip\cdot x}\Big]_{p^0 = E_{\vec{p}}} \,.
\end{aligned}$$

As long as $p^2 = m^2$ as required by $p^0 = E_{\vec{p}}$, this is a solution. Unlike in the real case where we needed $\phi = \phi^*$, a and b can now be chosen independently.

Homework 2: Scalar electrodynamics (12 Pts.)

Consider the following Lagrangian

$$\mathcal{L} = (D_\mu\phi)(D^\mu\phi)^* - m^2\phi\phi^* - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} \tag{4}$$

with the covariant derivative

$$D_\mu = (\partial_\mu - ieA_\mu) \tag{5}$$

and field-strength tensor

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu\,. \tag{6}$$

a) (4 Pts.) Expand $\mathcal{L}$ and identify the electromagnetic current j_μ and photon mass m_A^2 in terms of the field ϕ

$$\mathcal{L} \supset A^\mu j_\mu - m_A^2 A_\mu A^\mu - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}\,. \tag{7}$$

b) (8 Pts.) Derive the Maxwell equation as an Euler-Lagrange equation for A_μ

$$\frac{\partial}{\partial x_\mu}\frac{\partial\mathcal{L}}{\partial(\partial^\mu A_\nu)} - \frac{\partial\mathcal{L}}{\partial A_\nu} = 0 \tag{8}$$

How is this similar or different from what you are used to?

SOLUTION:

a)

$$\mathcal{L} = \partial_\mu\phi\partial^\mu\phi^* + A^\mu \underbrace{\left(ie(\phi^*\partial_\mu\phi - \phi\partial_\mu\phi^*)\right)}_{j_\mu} - m^2\phi\phi^* + \underbrace{e^2\phi\phi^*}_{-m_A^2} A_\mu A^\nu - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}$$

with

$$m_A^2 = -e^2\phi\phi^*$$
$$j_\mu = ie(\phi^*\partial_\mu\phi - \phi\partial_\mu\phi^*)$$

b) We write

$$\frac{\partial\mathcal{L}}{\partial(\partial_\mu A_\nu)} = -F_{\mu\nu} \qquad \text{and} \qquad \frac{\partial\mathcal{L}}{\partial A_\nu} = j_\mu - 2m_A^2 A_\mu$$

and therefore the Euler Lagrange equation

$$\frac{\partial}{\partial x_\mu}F_{\mu\nu} = -j_\mu + 2m_A^2 A_\mu\,.$$

In the limit of $m_A \to 0$, this corresponds the classical electrodynamics.