MATH425
Quantum Field Theory
Homework Sheet 0

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Due: never

This bonus sheet is a collection of problems you may find useful. Please consider these questions in addition to the exercises in the lecture notes.

Note: the content covered here is not necessarily the same as in the exam.

Homework 1: Quantum harmonic oscillator

The Hamiltonian for the quantum harmonic oscillator is given by $\hat{H} = \frac{\hat{p}^2}{2m} + \frac{m\omega^2}{2}\hat{q}^2$.

a) (2 Pts.) Using the fact that $[\hat{q}, \hat{p}] = i$, rewrite the Hamiltonian with the raising and lowering operators $\hat{a}$ and $a^\dagger$ that have the communtation relations $[a, a^\dagger] = 1$.

b) (2 Pts.) What are the commutation relation of the operator and $N = a^\dagger a$ with the Hamiltonian operator?

c) (2 Pts.) With the help of the a and $a^\dagger$ operators, construct the spectrum of energy eigenstates of the Hamiltonian starting from the vacuum state $|0\rangle$.

d) (1 Pts.) Find corresponding energy levels?

e) (1 Pts.) What is the energy level of the vacuum state $|0\rangle$? How would you interpret this as in implication for observable phenomena?

SOLUTION:

a) The Hamiltonian for a quantum harmonic oscillator is

$$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega^2\hat{q}^2 \,.$$

We define the raising and lowering operators $\hat{a}$ and $\hat{a}^\dagger$ as

$$\hat{q} = \sqrt{\frac{}{2m\omega}}(\hat{a} + \hat{a}^\dagger), \quad \hat{p} = i\sqrt{\frac{m\omega}{2}}(\hat{a}^\dagger - \hat{a}) \,.$$

Substituting these into the Hamiltonian, we find:

$$\hat{H} = \omega\left(\hat{a}^\dagger\hat{a} + \frac{1}{2}\right) .$$

b) The number operator is $\hat{N} = \hat{a}^\dagger\hat{a}$. The commutation relations are

$$[\hat{H}, \hat{a}] = -\omega\hat{a}, \quad [\hat{H}, \hat{a}^\dagger] = \omega\hat{a}^\dagger \,.$$

and for the number operator

$$[\hat{N}, \hat{a}] = -\hat{a}, \quad [\hat{N}, \hat{a}^\dagger] = \hat{a}^\dagger \,.$$

c) Define the vacuum state $|0\rangle$ such that $\hat{a}|0\rangle = 0$. The excited states are obtained by applying $\hat{a}^\dagger$

$$|n\rangle = \frac{(\hat{a}^\dagger)^n}{\sqrt{n!}}|0\rangle \,.$$

d) The energy levels of the oscillator are

$$E_n = \omega\left(n + \frac{1}{2}\right), \quad n = 0, 1, 2, \ldots \,.$$

e) The energy of the vacuum state $|0\rangle$ is

$$E_0 = \frac{1}{2}\omega \,.$$

This is the zero-point energy, an inherent energy due to quantum fluctuations. It implies particles cannot have zero energy, contributing to phenomena such as the Casimir effect.

Homework 2: Harmonic oscillator and Euler-Lagrange equations The potential function of a one dimensional harmonic oscillator is given by

$$V(x) = \frac{1}{2}kx^2.$$

a) (1 Pts.) Write the Lagrangian $L = T - V$, where T is the kinetic energy and V is the potential energy.

b) (1 Pts.) Use the Euler-Lagrange equation to derive the equation of motion.

c) (2 Pts.) Using the initial conditions $x(0) = 0$ and $\dot{x}(0) = v_0$, find the specific solution for $x(t)$.

d) (2 Pts.) Write expressions for the kinetic energy $T(t)$, potential energy $V(t)$, and the total energy E of the system. Prove that the total energy is conserved over time.

e) (2 Pts.) Suppose a damping force proportional to velocity is added to the system, such that the equation of motion becomes $m\ddot{x} + b\dot{x} + kx = 0$ where b is the damping coefficient. Solve the new equation of motion for the underdamped case (where $b^2 < 4mk$) and describe how the motion differs from the undamped case.

SOLUTION:

a) The potential is $V(x) = \frac{1}{2}kx^2$ and the kinetic energy $T = \frac{1}{2}m\dot{x}^2$. Therefore, the Lagrangian is

$$L = T - V = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}kx^2 \,.$$

b) Using the Euler-Lagrange equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = 0 \,.$$

we find

$$m\ddot{x} + kx = 0 \,.$$

c) With initial conditions $x(0) = 0$ and $\dot{x}(0) = v_0$, the solution to $m\ddot{x} + kx = 0$ is

$$x(t) = \frac{v_0}{\omega}\sin(\omega t), \quad \text{where } \omega = \sqrt{\frac{k}{m}} \,.$$

d) The kinetic energy is $T(t) = \frac{1}{2}m\dot{x}^2$ and potential energy is $V(t) = \frac{1}{2}kx^2$. Using $x(t)$ from above, we find that the total energy $E = T + V$ remains constant over time, indicating conservation of energy.

e) For the damped equation $m\ddot{x} + b\dot{x} + kx = 0$ with $b^2 < 4mk$, the solution is

$$x(t) = e^{-\gamma t}\left(C_1\cos(\omega' t) + C_2\sin(\omega' t)\right) \,.$$

where $\gamma = \frac{b}{2m}$ and $\omega' = \sqrt{\omega^2 - \gamma^2}$. This represents oscillatory motion with a decreasing amplitude due to damping.

To verify, substitute $x(t) = e^{-\gamma t}(C_1\cos(\omega' t) + C_2\sin(\omega' t))$ into the differential equation.

(a)

$$\dot{x}(t) = -\gamma e^{-\gamma t}\left(C_1\cos(\omega' t) + C_2\sin(\omega' t)\right) + e^{-\gamma t}\left(-C_1\omega'\sin(\omega' t) + C_2\omega'\cos(\omega' t)\right)$$

(b)

$$\ddot{x}(t) = e^{-\gamma t}\left((\gamma^2 - \omega'^2)(C_1\cos(\omega' t) + C_2\sin(\omega' t)) - 2\gamma\omega'(-C_1\sin(\omega' t) + C_2\cos(\omega' t))\right)$$

Substituting $\ddot{x}(t)$, $\dot{x}(t)$, and $x(t)$ back into the equation $m\ddot{x} + b\dot{x} + kx = 0$, we find each term balances, confirming that this is a solution. The underdamped motion is oscillatory with a decaying amplitude due to the factor $e^{-\gamma t}$.

Homework 3: Real and complex scalar fields and dark matter scattering

The interaction Lagrangian in a theory with a real scalar field ϕ, and a complex scalar field ψ is given by:

$$\mathcal{L}_I = y\,\phi\,\psi^\dagger\psi\,.$$

a) (1 Pts.) Use the Feynman diagram technique to find the decay amplitude for the field ϕ.

b) (3 Pts.) Compute the decay rate Γ for process field $\phi \to \psi^\dagger\psi$.

c) (1 Pts.) For which value of m_ψ is the decay rate maximal?

d) (2 Pts.) What is the highest value of m_ψ for which the decay rate is well defined? What is the physical interpretation of this fact?

We can also view this as a simplified model of Dark Matter scattering

e) (1 Pts.) Using the Feynman diagram technique, write down the amplitude for $\psi^\dagger(p_1)+\psi(p_2) \to \phi(q_1)+\phi(q_2)$.

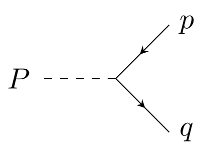
f) (4 Pts.) Compute the corresponding total annihilation cross section in the limit that $m_\phi = 0$, and the relative velocity of ψ and $\psi^\dagger$ is low.

g) (1 Pts.) What the annihilation rate of ψ and $\psi^\dagger$, in that case that their number densities are given by $n = n_\psi = n_{\psi^\dagger}$?

SOLUTION:

a) We have a single diagram for the process $\phi \to \psi^\dagger\psi$

$\mathcal{M} =$



$= -iy\,.$

b) From the lecture notes, we have

$$\mathrm{d}\Gamma = \frac{1}{2m_\phi}\mathrm{d}\Phi_{1\to 2}|\mathcal{M}|^2 = \frac{1}{2m_\phi}\mathrm{d}\Omega\frac{1}{16\pi^2}\frac{|\vec{p}|}{E_{\mathrm{cm}}}y^2\,.$$

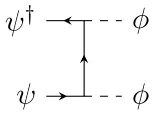
Setting $E_{\mathrm{cm}} = m_\phi$ and $m_\psi^2 = (m_\phi/2)^2 - |\vec{p}|^2$ we have

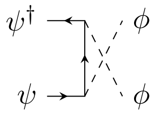
$$\Gamma = \frac{y^2}{16\pi m_\phi}\sqrt{1-\frac{4m_\psi^2}{m_\phi^2}}\,.$$

c) Clearly $m_\psi = 0$ has $\Gamma = y^2/(16\pi m_\phi)$.

d) If $m_\phi < 2m_\psi$ the process becomes inaccessible as the two ψ particles cannot be produced.

e) The Feynman diagrams are

$\mathcal{M} =$ 

$+$ 

$= (-iy)^2 \left(\frac{1}{(p_1 - q_1)^2 - m_\psi^2} + \frac{1}{(p_1 - q_2)^2 - m_\psi^2} \right)$

f)

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{1}{32\pi^2} \frac{|\mathcal{M}|^2}{4s} \frac{p_f}{p_i}$$

The amplitude is angle independent, so the $d\Omega$ integral can be evaluated and yields a factor 4π. Now, in the limit of small v_{rel}, the kinematic variables can be expanded as

$$s \approx 4m_\psi^2 \text{ and } p_i \approx v_{\mathrm{rel}} m_\psi \text{ and } p_f \approx m_\psi \,.$$

With $(p_1 - q_1)^2 \approx (p_1 - q_2)^2 \approx -m_\psi^2$ the matrix element is also expanded

$$\mathcal{M} \approx \frac{y^2}{m_\psi^2} \,.$$

Thus we have (including the symmetry factor for identical final state particles)

$$\sigma = \frac{1}{128\pi} \frac{1}{v_{\mathrm{rel}}\, m_\psi^2} \frac{y^4}{m_\psi^4} \,.$$

g) The annihilation rate is given by

$$\Gamma_{\mathrm{ann}} = \sigma n_\psi v_{\mathrm{rel}} = \frac{n_\psi}{128\pi} \frac{1}{m_\psi^2} \frac{y^4}{m_\psi^4}$$

Homework 4: Renormalisation

Consider the bare Lagrangian

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi_0)(\partial^\mu \phi_0) - \frac{1}{2}m_0^2\phi_0^2 - \frac{\lambda_0}{3!}\phi_0^4$$

of a real scalar quantum field theory.

a) (10 Pts.) Calculate the 1PI contributions to $\langle \Omega | T\{\phi\phi\} | \Omega \rangle$ and $\langle p_3, p_4 | S | p_1, p_2 \rangle$ for vanishing momenta.

b) (2 Pts.) Derive the counterterm Lagrangian by replacing the bare parameter $\phi_0 \to Z_\phi^{1/2}\phi$, $m_0 \to Z_m m$ and $\lambda_0 \to Z_\lambda \lambda_0$ in terms of renormalised parameters and separating the counterterm Lagrangian from the renormalised Lagrangian.

c) (5 Pts.) Express the degree of divergence of a diagram as a function of the number of vertices V and external lines N_e.

d) (2 Pts.) For which N_e are diagrams finite after all subdivergences are subtracted?

e) (2 Pts.) Explain if this theory is renormalisable.

SOLUTION: See lecture notes.

a) cf. eqs. (414) and (418) or (423) and (424)

b) cf. eq. (428)

c) cf. eq. (454)

d) cf. Figure 8

e) cf. Section 7.4.1

Homework 5: Euler-Lagrange equations for $\phi^3+\phi^4$

Consider the Lagrangian density $\mathcal{L}=\mathcal{L}_0+\mathcal{L}_I$, with

$$\mathcal{L}_0=\frac{1}{2}(\partial_\mu\phi)(\partial^\mu\phi)-\frac{1}{2}m^2\phi^2 \qquad \text{and} \qquad \mathcal{L}_I=-\frac{\lambda}{3!}\phi^3-\frac{\lambda}{4!}\phi^4\,.$$

a) (6 Pts.) Derive the equation of motion from

$$\partial^\mu\left[\frac{\partial\mathcal{L}}{\partial(\partial^\mu\phi)}\right]-\frac{\partial\mathcal{L}}{\partial\phi}=0.$$

b) (4 Pts.) With the generalized momentum $\pi(x)=(\partial\mathcal{L})/(\partial\dot{\phi})$ write down the Hamiltonian defined by

$$H=\int \mathrm{d}^3x[\pi\dot{\phi}-\mathcal{L}]\,.$$

c) (10 Pts.) Using the canonical commutation relations, show that $i\dot{\pi}=[\pi,H]$.

SOLUTION:

a) Using $(\partial_\mu\phi)(\partial^\mu\phi)=\eta_{\rho\sigma}\partial^\rho\phi\partial^\sigma\phi$, we have

$$\frac{\partial}{\partial(\partial^\mu\phi)}(\partial\phi)^2=2\partial_\mu\phi\,.$$

Therefore,

$$\partial^\mu\partial_\mu\phi=-m^2\phi-\frac{1}{2!}\lambda_3\phi^2-\frac{1}{3!}\lambda_4\phi^3$$

b) The generalised momentum is unchanged from the free case $\pi=\dot{\phi}$, so

$$H=\int \mathrm{d}^3x\Big[\frac{1}{2}\dot{\phi}^2+\frac{1}{2}(\nabla\phi)^2+\frac{1}{2}m^2\phi^2+\frac{1}{3!}\lambda_3\phi^3+\frac{1}{4!}\lambda_4\phi^4\,.$$

c) Using commutation algebra we can write

$$[\pi(\vec{x}),\phi(\vec{y})^3]=\phi(\vec{y})[\pi(\vec{x}),\phi(\vec{y})^2]+[\pi(\vec{x}),\phi(\vec{y})]\phi(\vec{y})^2=-3i\phi(\vec{y})^2\;\delta(\vec{x}-\vec{y})\,.$$
$$[\pi(\vec{x}),\phi(\vec{y})^4]=-4i\phi(\vec{y})^3\;\delta(\vec{x}-\vec{y})\,.$$

We have already shown that

$$[\pi(\vec{x}),H_0]=i(\nabla^2\phi(\vec{x})-m^2\phi(\vec{x}))$$

and now can show that

$$[\pi(\vec{x}),H_I]=-i\int\left[\lambda_3\frac{1}{2}\delta(\vec{x}-\vec{y})\phi(\vec{y})^2+\lambda_4\frac{1}{3!}\phi(\vec{y})^3\right]\mathrm{d}^3y\,.$$

Therefore,

$$[\pi(\vec{x}),H]=i\left[\nabla^2\phi(\vec{x})-m^2\phi(\vec{x})-\frac{1}{2!}\lambda_3\phi(\vec{x})^2-\frac{1}{3!}\lambda_3\phi(\vec{x})^3\right]=i\dot{\pi}(\vec{x})\,.$$

Homework 6: Dirac field (6 Pts.)

The Dirac field may be written

$$\psi(x) = \int \frac{\mathrm{d}^3 p}{(2\pi)^3 \sqrt{2E_{\vec{p}}}} \sum_{s=1}^{2} \left[\psi_p^{(s)}(x) a_s(\vec{p}) + \tilde{\psi}_p^{(s)}(x) b_s^\dagger(\vec{p}) \right],$$

where

$$\psi_p^{(s)}(x) = e^{-ip\cdot x} u_s(p)\,,$$
$$\tilde{\psi}_p^{(s)}(x) = e^{ip\cdot x} v_s(p)\,.$$

With the scalar product $\langle \psi_1 | \psi_2 \rangle = \int \mathrm{d}^3 x\ \psi_1^\dagger \psi_2$, show that

$$\langle \psi_p^{(r)} | \psi_q^{(s)} \rangle = \delta_{rs} (2\pi)^3 2E_{\vec{p}}\, \delta^{(3)}(\vec{p} - \vec{q}).$$

SOLUTION:

$$\langle \psi_p^{(r)} | \psi_q^{(s)} \rangle = \int \mathrm{d}^3 x \psi_p^{(r)\dagger} \psi_q^{(s)} = e^{i(p^0 - q^0)t} \int \mathrm{d}^3 x e^{-i(\vec{p}-\vec{q})\cdot\vec{x}} \psi_p^{(r)\dagger} \psi_q^{(s)}$$

Homework 7: Wick theorem

Use the Wick theorem to calculate the symmetry factors of the diagrams in (190), (193), (210), (211), (212), (213), (214) of the lecture notes.

SOLUTION: See lecture notes.