MATH425
Quantum Field Theory
Homework Sheet 2

Academic Year 2025/26
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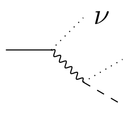
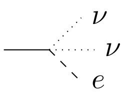
Homework 1: A simple model of the muon decay (10 Pts.)

Context: A muon is a lepton like the electron but its mass is approximately 200× larger ($m_e = 0.511\,\text{MeV}$ vs. $m_\mu = 105\,\text{MeV}$). In the Standard Model of particle physics, the muon can decay into an electron and two neutrinos, i.e. $\mu \to e\nu\bar\nu$. This decay is mediated by a W boson

$\mu \to e\nu\bar\nu = \mu$  . (1)

One can approximate this amplitude by replacing the W with a four-point vertex

$\mu \to e\nu\bar\nu \approx \mu$  (2)

In this exercise sheet we will simplify this further and treat the muon, electron and neutrino as scalar particle ϕ_1, ϕ_2, ϕ_3 respectively.

Consider the following theory

$$\mathcal{L} = \sum_{i=1}^{3} \left[\frac{1}{2}(\partial_\mu\phi_i)(\partial^\mu\phi_i) - \frac{1}{2}m_i^2\phi_i^2\right] - \frac{1}{2!}\lambda\,\phi_1\phi_2\phi_3^2\,, \tag{3}$$

with $m_3 \equiv m_\nu = 0$.

a) (2 Pts.) Write down the Feynman rules of this theory. You do not need to derive these from first principle.

b) (1 Pts.) Calculate the amplitude for the process $\phi_1(P) \to \phi_2(p_1)\phi_3(p_2)\phi_3(p_3)$ as a proxy for $\mu \to e\nu\bar\nu$.

To calculate the total decay rate, we can make use of the fact that if we have a multi-particle phase space like $\mathrm{d}\Phi_3$, we can write this as

$$\mathrm{d}\Phi_3(\phi_1 \to \phi_2\phi_3\phi_3) = \mathrm{d}\Phi_2\Big(\phi_1(P) \to \phi_2(p_1)Q^2\Big)\mathrm{d}\Phi_2\Big(Q^2 \to \phi_3(p_2)\phi_3(p_3)\Big)\frac{1}{2\pi}\mathrm{d}Q^2\,. \tag{4}$$

Here we have introduced a new parameter Q^2 which we can view as a fictitious particle that mediates the process with $Q = p_2 + p_3$. We will now calculate the differential decay rate $\mathrm{d}\Gamma/\mathrm{d}Q^2$.

c) (2 Pts.) To do this, go first into the restframe of Q^2 where $\vec p_2^{\,*} = -\vec p_3^{\,*}$ and calculate

$$\int \mathrm{d}\Phi_2(Q^2 \to \phi_3\phi_3)|\mathcal{M}|^2\,. \tag{5}$$

d) (2 Pts.) Calculate $|\vec{p}_1| = \sqrt{\Lambda}/(2m_1)$ to find Λ in terms of m_1, m_2, and Q^2.

e) (2 Pts.) Calculate $\mathrm{d}\Gamma/\mathrm{d}Q^2$.

f) (1 Pts.) Set $m_2 = 0$ and calculate Γ.