MATH425
Quantum Field Theory
Homework Sheet 1

UNIVERSITY OF LIVERPOOL

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Homework 1: Complex scalar field (8 Pts.)

Consider the following Lagrangian of the complex scalar field

$$\mathcal{L} = (\partial_\mu \phi)(\partial^\mu \phi)^* - m^2 \phi\phi^* \,. \tag{1}$$

Compared to the real scalar field we now have two *independent* fields ϕ and ϕ^*.

a) (2 Pts.) Derive the equations of motion for $\phi(x)$ and $\phi^*(x)$.

b) (2 Pts.) Write down the conjugate momenta $\pi(x)$ and $\pi^*(x)$ as well as the Hamiltonian $\mathcal{H}$

c) (2 Pts.) Assume that $\phi(x)$ is a solution to the equations of motion to show that we have a conserved current in

$$j^\mu = -i(\phi\partial^\mu\phi^* - \phi^*\partial^\mu\phi) \,. \tag{2}$$

d) (2 Pts.) Explain why the general complex solution is given by

$$\phi(x) = \int \frac{\mathrm{d}^3 p}{(2\pi)^3} \sqrt{\frac{1}{2E_{\vec p}}} \Big[a_{\vec p}\, \mathrm{e}^{-ip\cdot x} + b^*_{\vec p}\, \mathrm{e}^{ip\cdot x}\Big]_{p^0=E_{\vec p}} \,, \tag{3}$$

where $a_{\vec p}$ and $b_{\vec p}$ are independent.

Homework 2: Scalar electrodynamics (12 Pts.)

Consider the following Lagrangian

$$\mathcal{L} = (D_\mu\phi)(D^\mu\phi)^* - m^2\phi\phi^* - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} \tag{4}$$

with the covariant derivative

$$D_\mu = (\partial_\mu - ieA_\mu) \tag{5}$$

and field-strength tensor

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \,. \tag{6}$$

a) (4 Pts.) Expand $\mathcal{L}$ and identify the electromagnetic current j_μ and photon mass m_A^2 in terms of the field ϕ

$$\mathcal{L} \supset A^\mu j_\mu - m_A^2 A_\mu A^\mu - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} \,. \tag{7}$$

b) (8 Pts.) Derive the Maxwell equation as an Euler-Lagrange equation for A_μ

$$\frac{\partial}{\partial x_\mu}\frac{\partial \mathcal{L}}{\partial(\partial^\mu A_\nu)} - \frac{\partial \mathcal{L}}{\partial A_\nu} = 0 \tag{8}$$

How is this similar or different from what you are used to?