MATH425
Quantum Field Theory
Homework Sheet 0

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Due: never

This bonus sheet is a collection of problems you may find useful. Please consider these questions in addition to the exercises in the lecture notes.

Note: the content covered here is not necessarily the same as in the exam.

Homework 1: Quantum harmonic oscillator

The Hamiltonian for the quantum harmonic oscillator is given by $\hat{H} = \frac{\hat{p}^2}{2m} + \frac{m\omega^2}{2}\hat{q}^2$.

a) (2 Pts.) Using the fact that $[\hat{q}, \hat{p}] = i$, rewrite the Hamiltonian with the raising and lowering operators $\hat{a}$ and $a^\dagger$ that have the communtation relations $[a, a^\dagger] = 1$.

b) (2 Pts.) What are the commutation relation of the operator and $N = a^\dagger a$ with the Hamiltonian operator?

c) (2 Pts.) With the help of the a and $a^\dagger$ operators, construct the spectrum of energy eigenstates of the Hamiltonian starting from the vacuum state $|0\rangle$.

d) (1 Pts.) Find corresponding energy levels?

e) (1 Pts.) What is the energy level of the vacuum state $|0\rangle$? How would you interpret this as in implication for observable phenomena?

Homework 2: Harmonic oscillator and Euler-Lagrange equations The potential function of a one dimensional harmonic oscillator is given by

$$V(x) = \frac{1}{2}kx^2.$$

a) (1 Pts.) Write the Lagrangian $L = T - V$, where T is the kinetic energy and V is the potential energy.

b) (1 Pts.) Use the Euler-Lagrange equation to derive the equation of motion.

c) (2 Pts.) Using the initial conditions $x(0) = 0$ and $\dot{x}(0) = v_0$, find the specific solution for $x(t)$.

d) (2 Pts.) Write expressions for the kinetic energy $T(t)$, potential energy $V(t)$, and the total energy E of the system. Prove that the total energy is conserved over time.

e) (2 Pts.) Suppose a damping force proportional to velocity is added to the system, such that the equation of motion becomes $m\ddot{x} + b\dot{x} + kx = 0$ where b is the damping coefficient. Solve the new equation of motion for the underdamped case (where $b^2 < 4mk$) and describe how the motion differs from the undamped case.

Homework 3: Real and complex scalar fields and dark matter scattering

The interaction Lagrangian in a theory with a real scalar field ϕ, and a complex scalar field ψ is given by:

$$\mathcal{L}_I = y\,\phi\,\psi^\dagger\psi\,.$$

a) (1 Pts.) Use the Feynman diagram technique to find the decay amplitude for the field ϕ.

b) (3 Pts.) Compute the decay rate Γ for process field $\phi \to \psi^\dagger\psi$.

c) (1 Pts.) For which value of m_ψ is the decay rate maximal?

d) (2 Pts.) What is the highest value of m_ψ for which the decay rate is well defined? What is the physical interpretation of this fact?

We can also view this as a simplified model of Dark Matter scattering

e) (1 Pts.) Using the Feynman diagram technique, write down the amplitude for $\psi^\dagger(p_1)+\psi(p_2) \to \phi(q_1) + \phi(q_2)$.

f) (4 Pts.) Compute the corresponding total annihilation cross section in the limit that $m_\phi = 0$, and the relative velocity of ψ and $\psi^\dagger$ is low.

g) (1 Pts.) What the annihilation rate of ψ and $\psi^\dagger$, in that case that their number densities are given by $n = n_\psi = n_{\psi^\dagger}$?

Homework 4: Renormalisation

Consider the bare Lagrangian

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\phi_0)(\partial^\mu\phi_0) - \frac{1}{2}m_0^2\phi_0^2 - \frac{\lambda_0}{3!}\phi_0^4$$

of a real scalar quantum field theory.

a) (10 Pts.) Calculate the 1PI contributions to $\langle\Omega|T\{\phi\phi\}|\Omega\rangle$ and $\langle p_3, p_4|S|p_1, p_2\rangle$ for vanishing momenta.

b) (2 Pts.) Derive the counterterm Lagrangian by replacing the bare parameter $\phi_0 \to Z_\phi^{1/2}\phi$, $m_0 \to Z_m m$ and $\lambda_0 \to Z_\lambda\lambda_0$ in terms of renormalised parameters and separating the counterterm Lagrangian from the renormalised Lagrangian.

c) (5 Pts.) Express the degree of divergence of a diagram as a function of the number of vertices V and external lines N_e.

d) (2 Pts.) For which N_e are diagrams finite after all subdivergences are subtracted?

e) (2 Pts.) Explain if this theory is renormalisable.

Homework 5: Euler-Lagrange equations for $\phi^3 + \phi^4$

Consider the Lagrangian density $\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_I$, with

$$\mathcal{L}_0 = \frac{1}{2}(\partial_\mu \phi)(\partial^\mu \phi) - \frac{1}{2}m^2\phi^2 \qquad \text{and} \qquad \mathcal{L}_I = -\frac{\lambda}{3!}\phi^3 - \frac{\lambda}{4!}\phi^4\,.$$

a) (6 Pts.) Derive the equation of motion from

$$\partial^\mu \left[\frac{\partial \mathcal{L}}{\partial(\partial^\mu \phi)}\right] - \frac{\partial \mathcal{L}}{\partial \phi} = 0.$$

b) (4 Pts.) With the generalized momentum $\pi(x) = (\partial \mathcal{L})/(\partial \dot{\phi})$ write down the Hamiltonian defined by

$$H = \int \mathrm{d}^3x[\pi\dot{\phi} - \mathcal{L}]\,.$$

c) (10 Pts.) Using the canonical commutation relations, show that $i\dot{\pi} = [\pi, H]$.

Homework 6: Dirac field (6 Pts.)

The Dirac field may be written

$$\psi(x) = \int \frac{\mathrm{d}^3p}{(2\pi)^3\sqrt{2E_{\vec{p}}}} \sum_{s=1}^{2} \left[\psi_p^{(s)}(x)a_s(\vec{p}) + \tilde{\psi}_p^{(s)}(x)b_s^\dagger(\vec{p})\right],$$

where

$$\psi_p^{(s)}(x) = e^{-ip\cdot x}u_s(p)\,,$$
$$\tilde{\psi}_p^{(s)}(x) = e^{ip\cdot x}v_s(p)\,.$$

With the scalar product $\langle\psi_1|\psi_2\rangle = \int \mathrm{d}^3x\ \psi_1^\dagger\psi_2$, show that

$$\langle\psi_p^{(r)}|\psi_q^{(s)}\rangle = \delta_{rs}(2\pi)^3 2E_{\vec{p}}\,\delta^{(3)}(\vec{p}-\vec{q}).$$

Homework 7: Wick theorem

Use the Wick theorem to calculate the symmetry factors of the diagrams in (190), (193), (210), (211), (212), (213), (214) of the lecture notes.